Rainbow Matchings of Size $\delta(G)$ in Properly Edge-Colored Graphs

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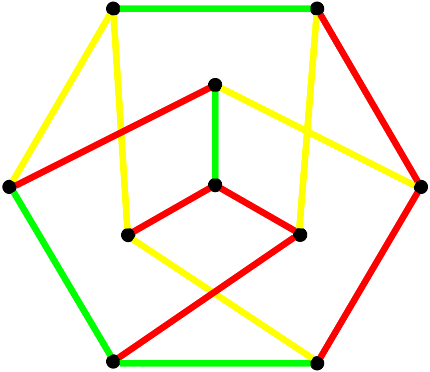

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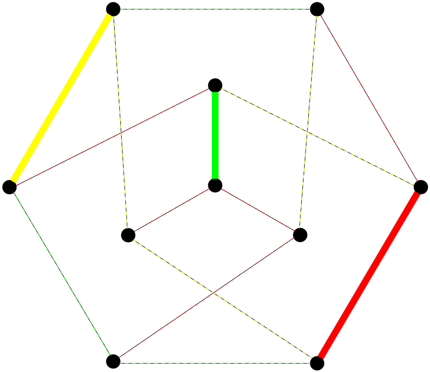

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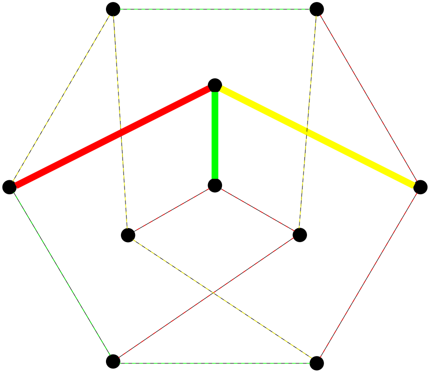

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Given a graph H, what conditions guarantee an edge-colored graph G contains a rainbow H?

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What conditions guarantee an edge-colored graph G contains a rainbow matching (of given size)?

Color Degree of v

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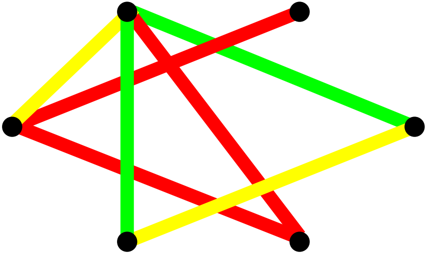

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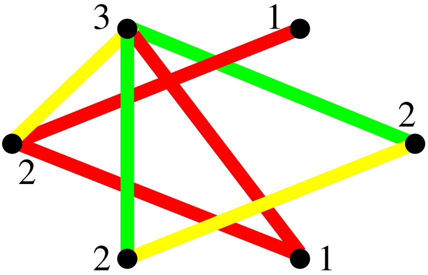

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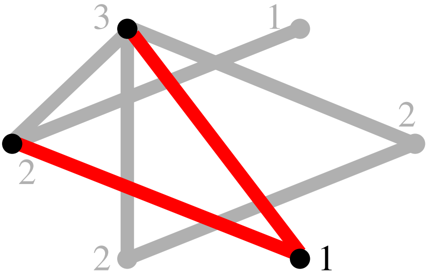

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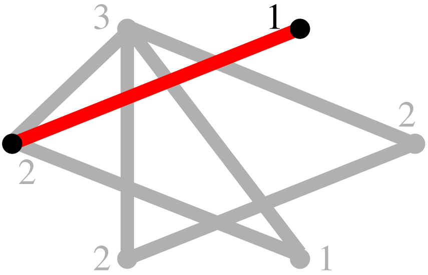

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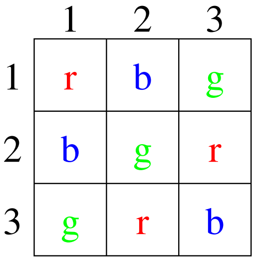

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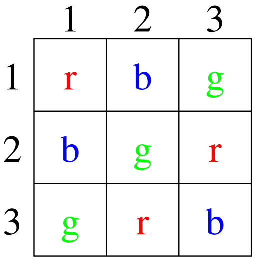

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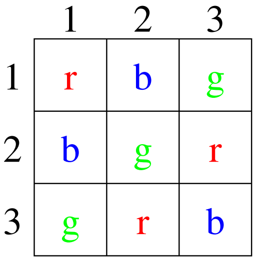

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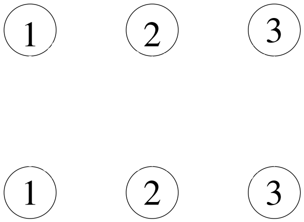

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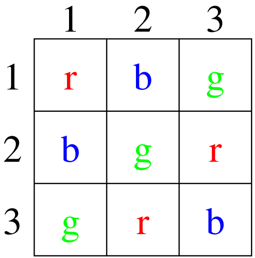

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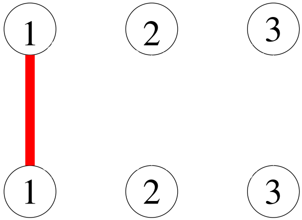

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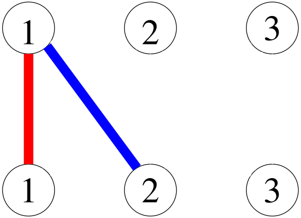

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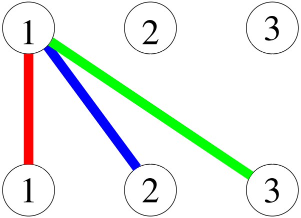

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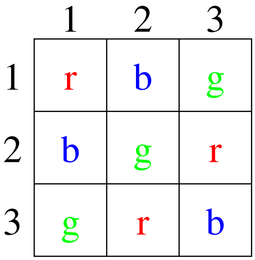

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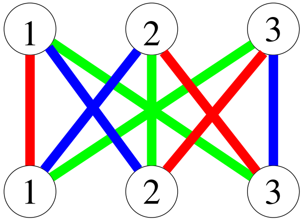

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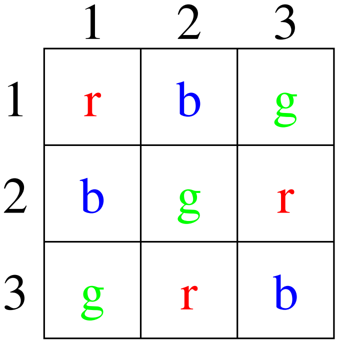

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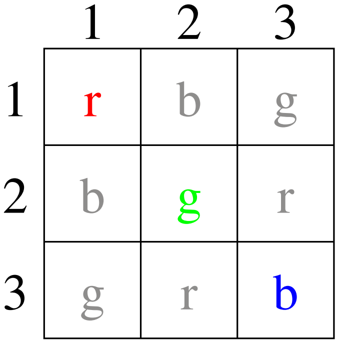

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Ryser's Conjecture

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Every Latin square of odd order has a transversal.

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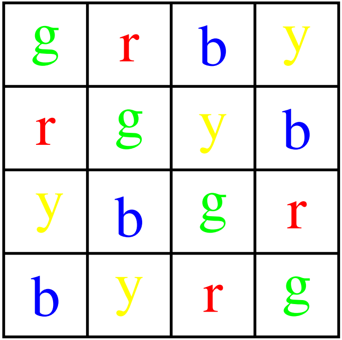

g	r	b	y
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Transversal Equivalent in $K_{n,n}$

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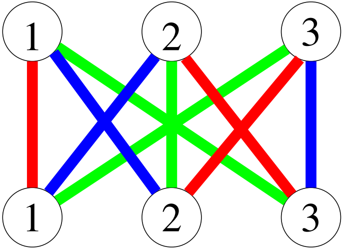

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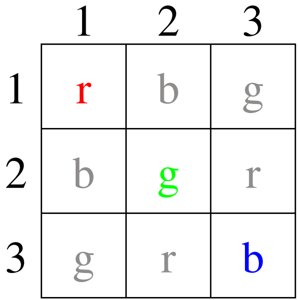

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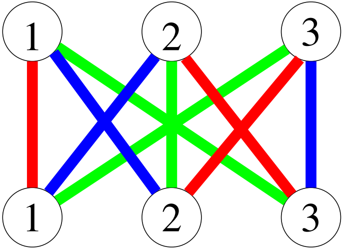

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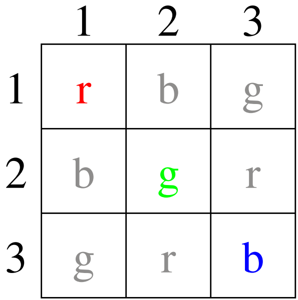

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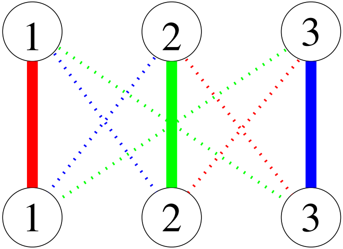

Transversals are rainbow perfect matchings in $K_{n,n}$!

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Theorem (Hatami, Shor, 2008)

Every proper edge-coloring of $K_{n,n}$ contains a rainbow matching of size $n - O(\log^2 n)$.

Theorem (Wang, 2011)

For a properly edge-colored graph G with $|V(G)| \geq 8\delta(G)/5$*, G has a rainbow matching of size at least* $\lfloor 3\delta(G)/5 \rfloor$*.*

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Main Problem

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$f(\delta) > 2\delta$ since there are even Latin squares without transversals.

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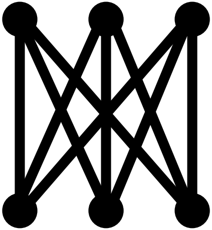

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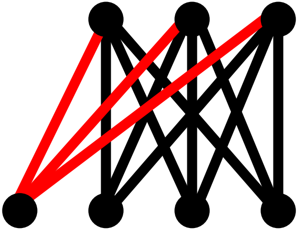

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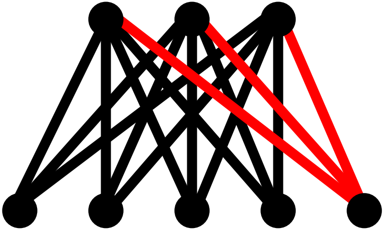

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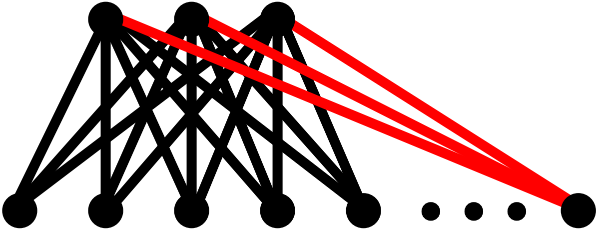

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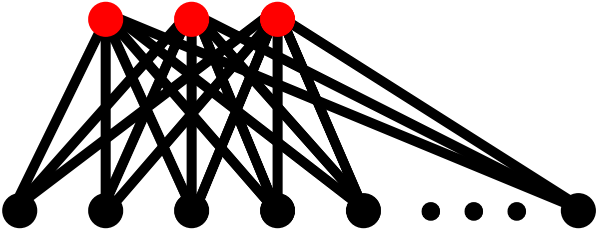

Bound on largest color class

Lemma

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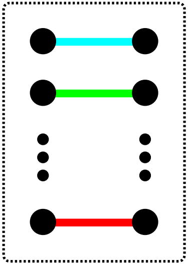

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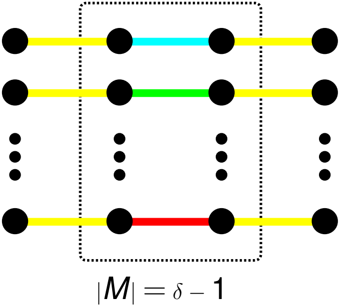

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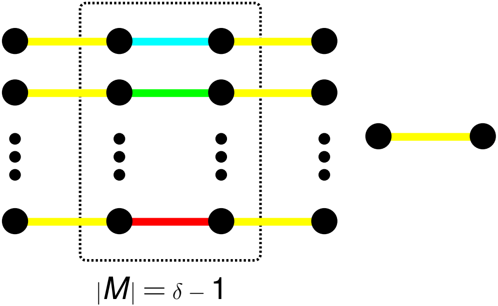

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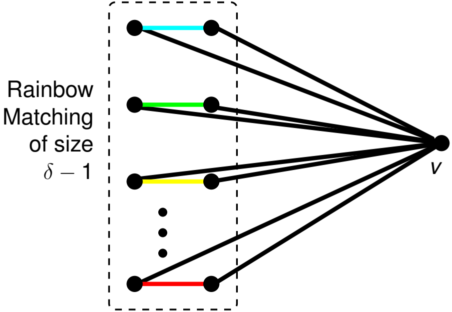

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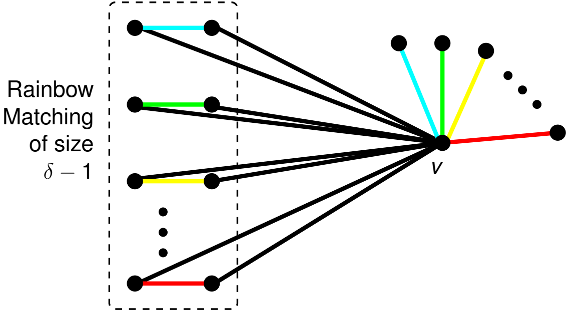

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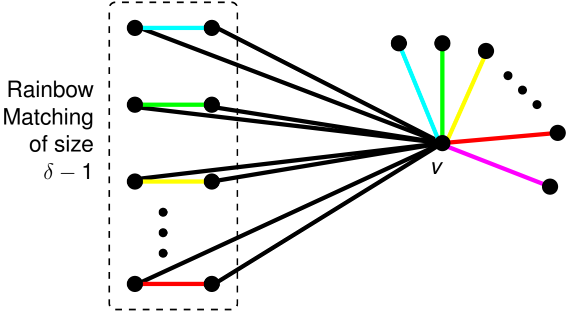

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In total, we delete at most 8δ edges in each iteration.

$$|V(G)| \geq 16\delta \Rightarrow |E(G)| \geq \frac{16\delta^2}{2} = 8\delta^2$$

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$$|V(G)| \geq 16\delta \Rightarrow |E(G)| \geq \frac{16\delta^2}{2} = 8\delta^2$$

Thus there are enough edges for δ iterations.

Our Results

Theorem (D., Ferrara, Lo, Moffatt, Pfender, Wenger, 2012+)

If G is a properly edge-colored graph with minimum degree $\delta = \delta(G)$ satisfying $|V(G)| \geq \frac{98}{23}\delta$, then G contains a rainbow matching of size $\delta(G)$.

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- Let G be a minimum counterexample, δ minimized.
- We can also assume that $|E(G)|$ is minimized, by preprocessing.

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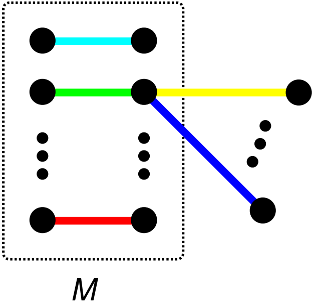

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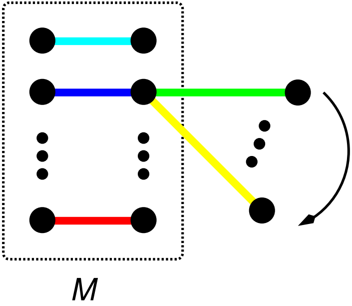

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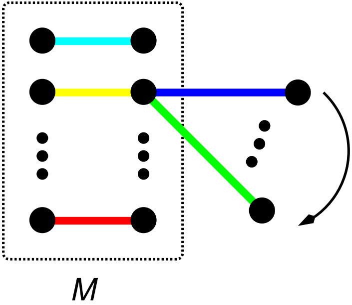

Levels of Flexibility

- **Flexible Edges** are edges in M whose pivot vertices see ‘enough’ edges with colors not used in the matching.

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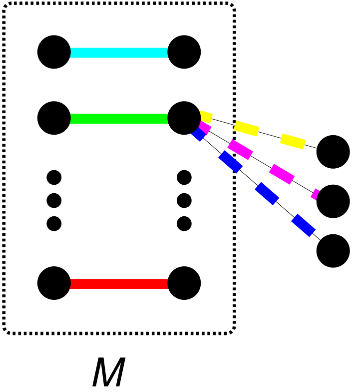

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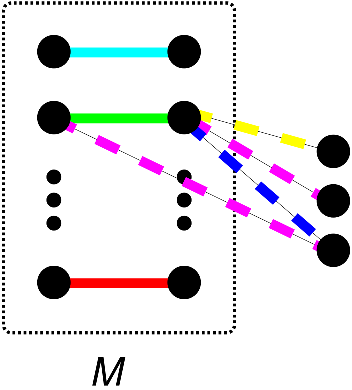

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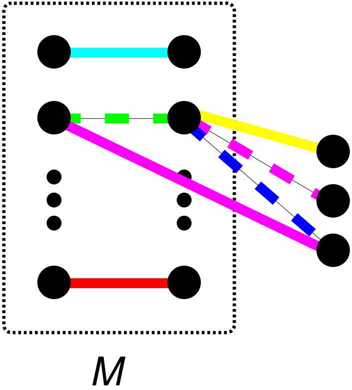

Bounding, Counting, and Algebra

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Under the assumption that M is maximal, $n < \frac{98}{23}\delta$, a contradiction.

Alternate $f(\delta)$, shown Algorithmically

Theorem (D., Ferrara, Lo, Moffatt, Pfender, Wenger, 2012+)

If G is a properly edge-colored graph with minimum degree δ satisfying

$$|V(G)| > \frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta},$$

then G has a rainbow matching of size δ, and there is an

$$O(\delta(G)|V(G)|^2)\text{-time}$$

algorithm that produces a rainbow matching of size δ in G.

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4. Stop when no edges remain in G.

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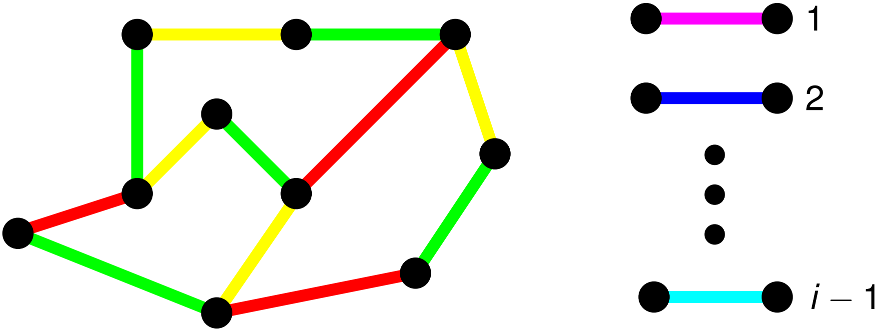

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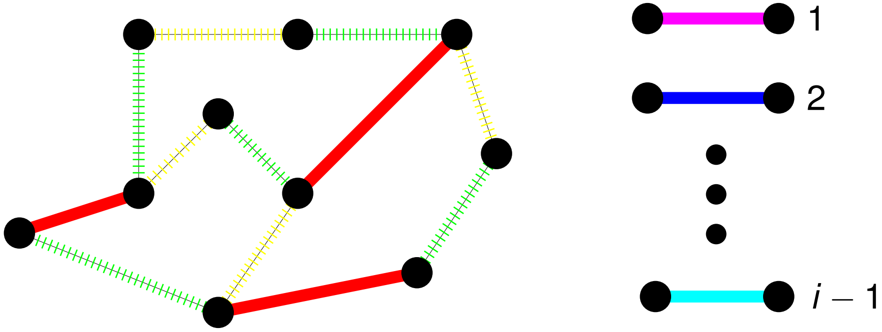

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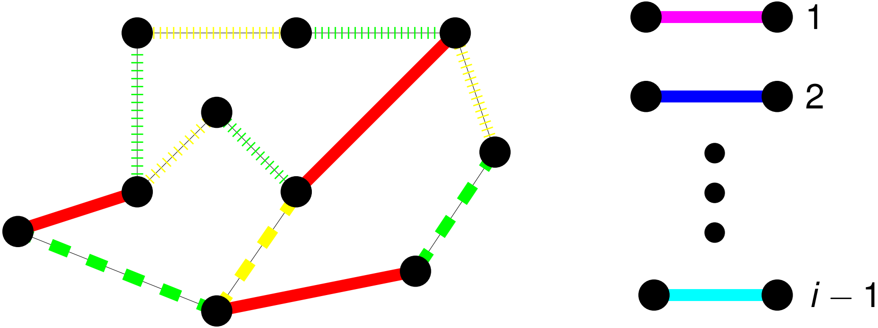

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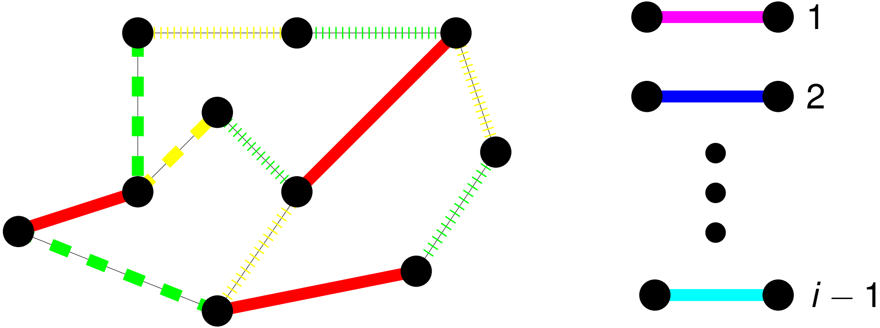

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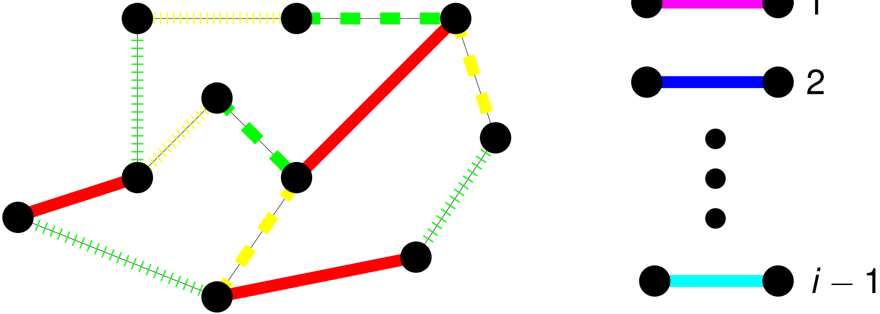

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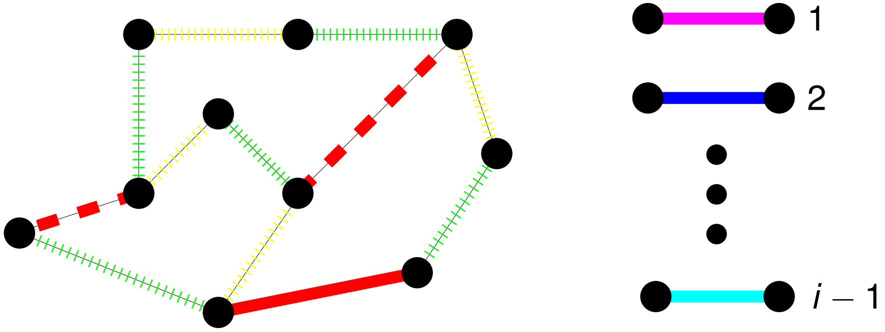

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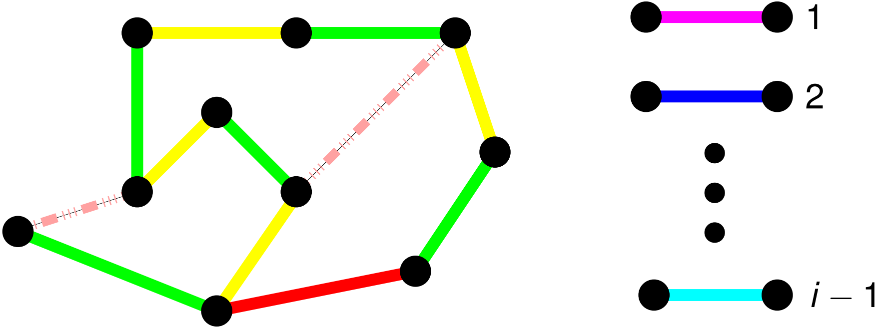

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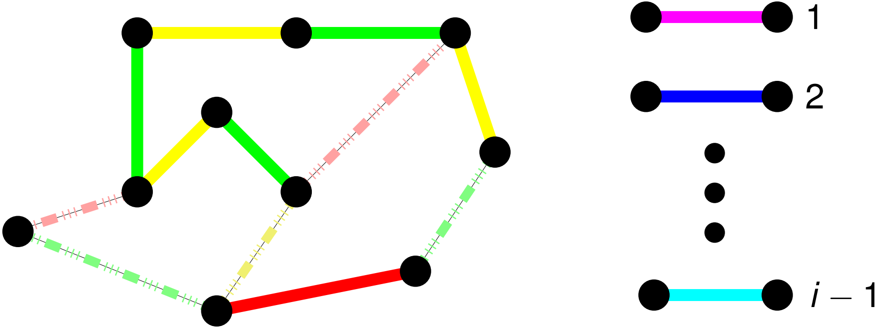
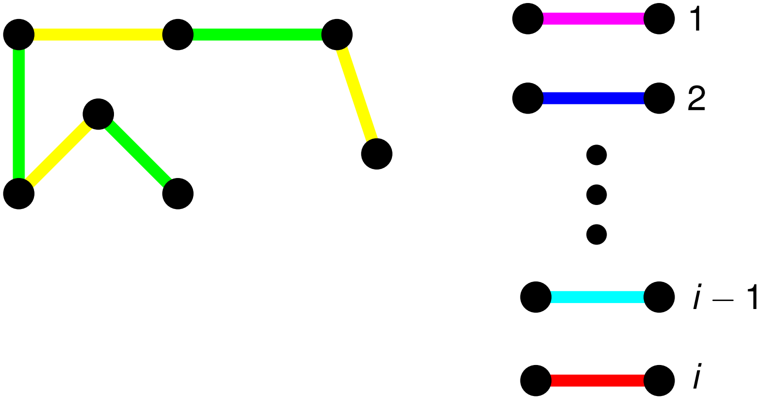


Add edge to the matching



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There are at most δ steps, thus $O(\delta|V(G)|^2)$.

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- Thank you!