



Locating a Robber on a Tree.

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Original Cops and Robbers

Pursuit-Evasion on a graph introduced by Parsons (1976).

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Game

- 1 *Cop chooses vertex*
- 2 *Robber chooses vertex*
- 3 *Cop and Robber take turns moving to adjacent vertices (visible to both players)*
- 4 *Cop wins if he 'arrests' the Robber*

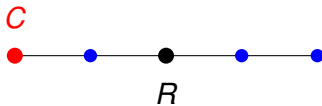
Examples



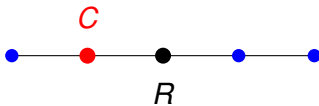
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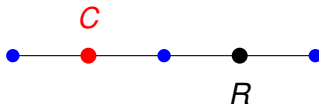
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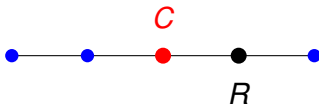
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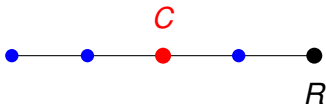
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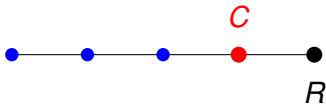
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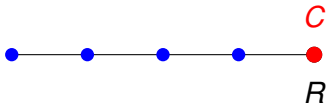
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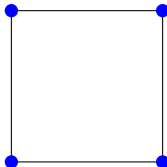
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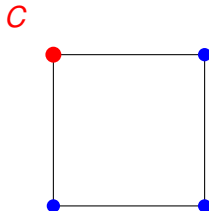
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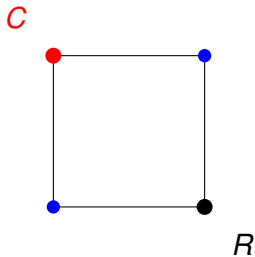
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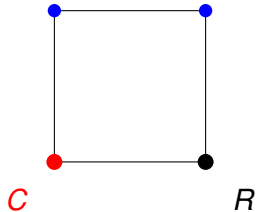
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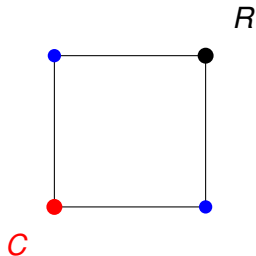
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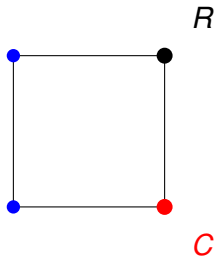
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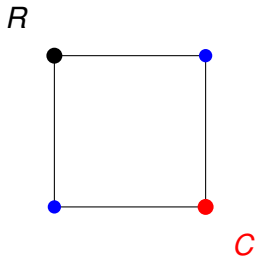
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Motivating Questions

Definition

A graph is ***cop-win*** if there is a strategy under which the Cop arrests the Robber in a finite number of moves. Otherwise, a graph is ***robber-win***.

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Example

A tree T is cop-win.

- 1 Which graphs are cop-win?
- 2 For a graph G , what is the minimum number of Cops required to capture the Robber?
- 3 What is the minimum guaranteed capture time?

Early Results

Theorem (Nowakowski, Winkler 1983)

A finite graph G is cop-win if and only if G is dismantlable.

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Theorem (Seymour, Thomas 1993)

A graph G has treewidth $k - 1$ iff k is the minimum number of cops for which G is cop-win.

On Large Classes

- Directed graphs (Hahn, MacGillivray 2006)
- Edge Critical graphs (Clarke, Fitzpatrick, Nowakowski 2010)
- Forbidden (induced) subgraphs (Joret, Kamiński, Theis 2010)
- Interval graphs (Gavenčiak 2011)
- Geometric graphs (Beveridge, Dudek, Frieze, Müller 2012)
- Random graphs (Scott, Sudakov 2011; Bollobas, Kun, Leader 2013)

Mobility Variations

- Robber can Hide and Ride (Chalopin, Chepoi 2011)
- Lazy Robber (Richerby, Thilikos 2011)
- Fast Robber (Alon, Mehrabian 2011; Frieze, Krivelevich, Loh 2012)
- Drunk Robber (Kehagias, Prałat 2012)

Search Variations

- Helicopter Search (Fomin 1998)
- Tandem Search (Clarke, Nowakowski 2005)
- Reduced Visibility (Isler, Karnad 2008)
- Witness (Clarke 2009)
- Fuel, Cost, Time (Fomin, Golovach, Prałat 2012)
- Distance Query (Seager 2012)

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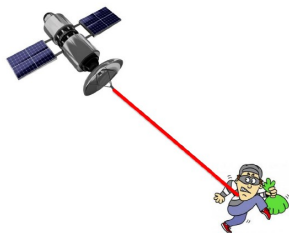
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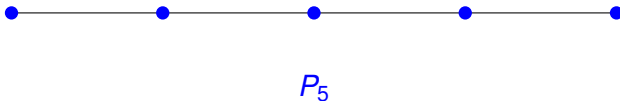
A graph G is **locatable** if the Cop has a winning strategy.

For locatable G , the **location number** of G , $loc(G)$, is the minimum number of probes guaranteed to locate the Robber.

Location Number of Paths

Proposition (Seager 2012)

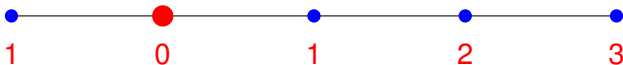
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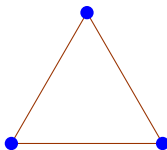
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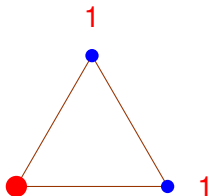
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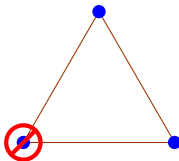
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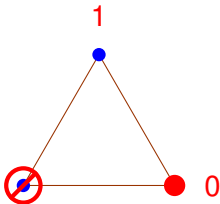
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If G contains $K_{3,3}$ as an induced subgraph, then G is not locatable.

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Proposition

C_n is locatable for $n > 5$. With, $\text{loc}(C_n) = 3$ for $6 \leq n \leq 11$ and $\text{loc}(C_n) = 2$ for $n > 11$.

Seager's Results For Trees

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If T is a tree with $n \geq 3$ vertices, then $\text{loc}(T) \leq n - 2$, with equality if and only if $T = K_{1,n-1}$.

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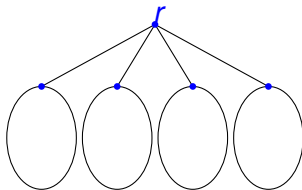
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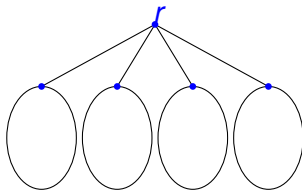


r-Locating Strategy

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Remark

Under an *r*-locating strategy, the Robber will never be able to move from one component of $T - r$ to another.

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- Improve bound for $\text{loc}(T)$ using parameters other than n

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d=4 Star of stars $\Rightarrow loc(T) < \ell - 1$.

d=5 One arm is star of stars, each other arms is a star $\Rightarrow < \ell - 1$.

Main Result

Theorem (Brandt, Diemunsch, Erbes, LeGrand, M. 2013+)

If T is a tree with $\text{diam}(T) = d \geq 6$ and $\Delta(T) = \Delta$, then

$$\text{loc}(T) \leq (2\Delta^2 - 6\Delta + 4)\left\lceil \frac{d}{2} \right\rceil - 4\Delta^2 + 13\Delta - 9.$$

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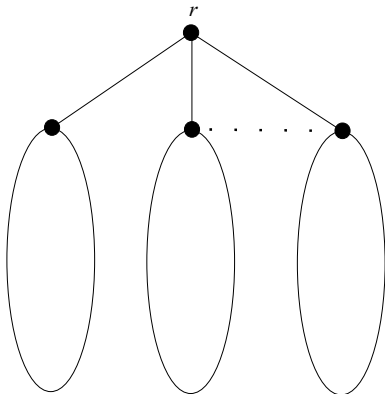
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Answering Seager's Questions

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- Conjecture $\text{loc}(T) \leq \ell - 1 \dots ?$

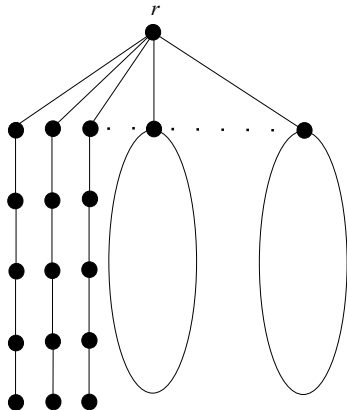
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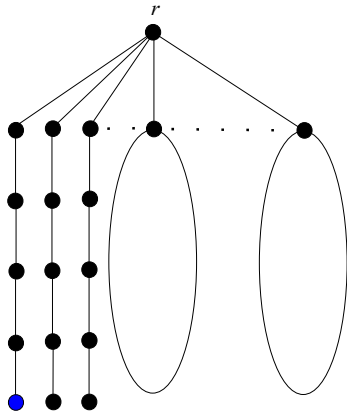
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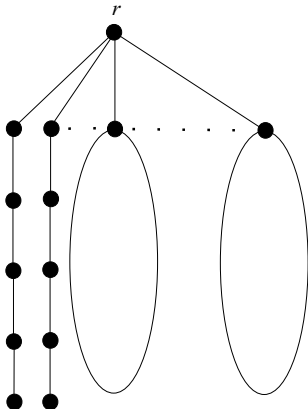
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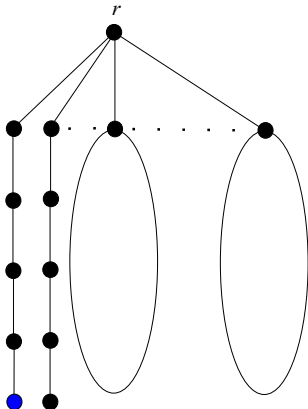
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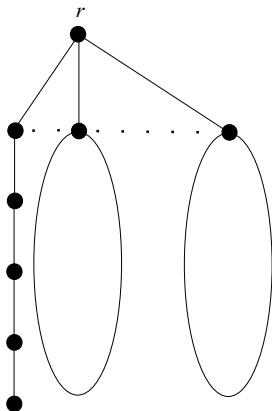
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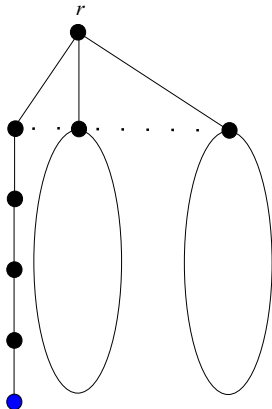
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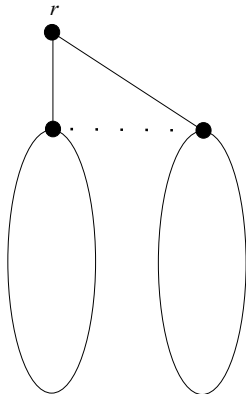
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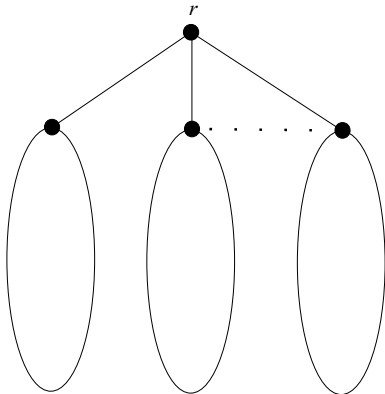
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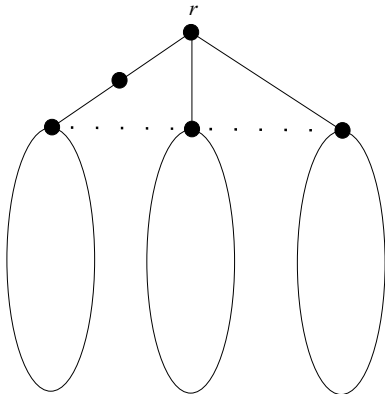
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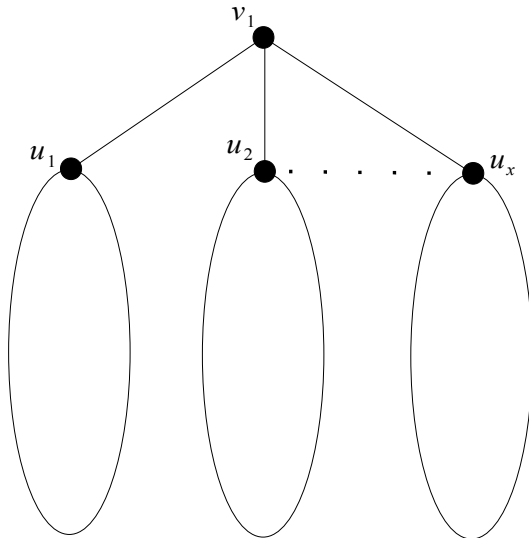


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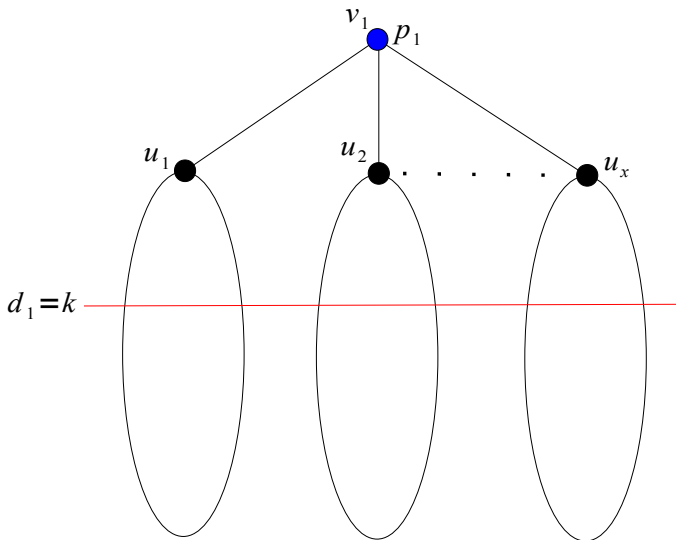
- Pick a Root.
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- Remove Long Arms.



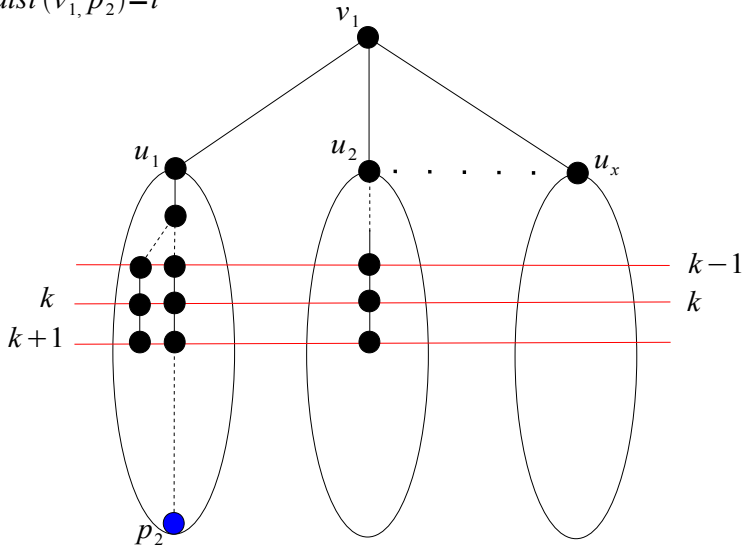
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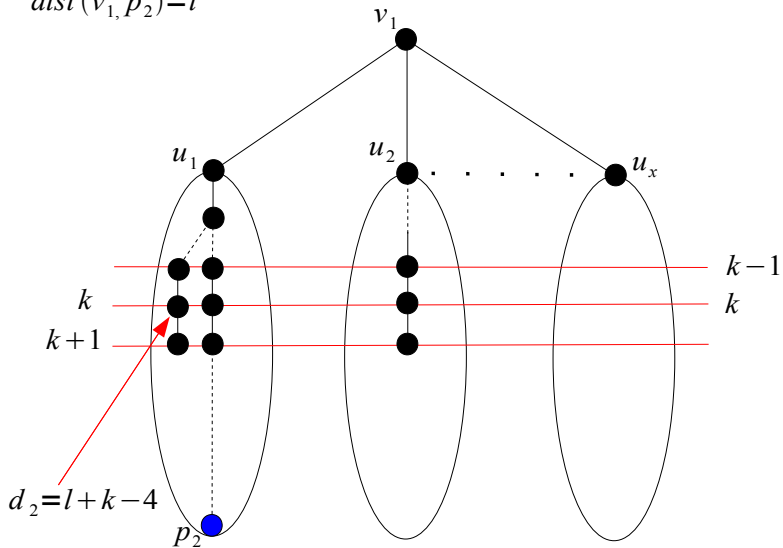


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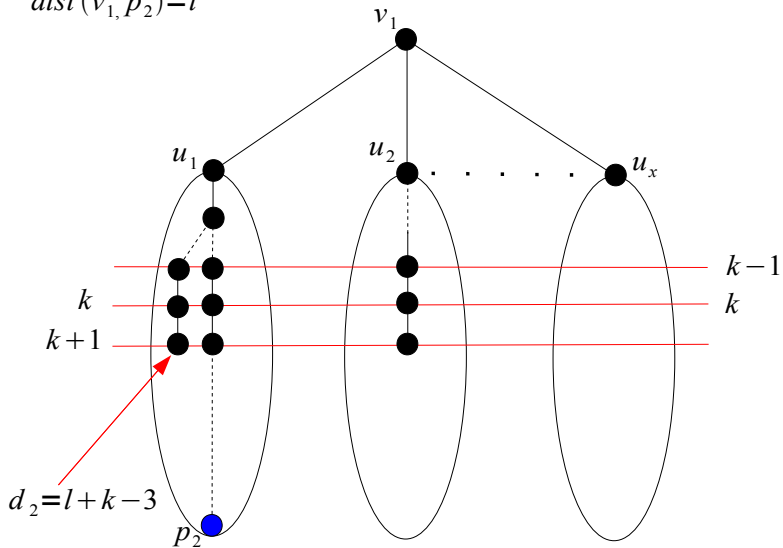
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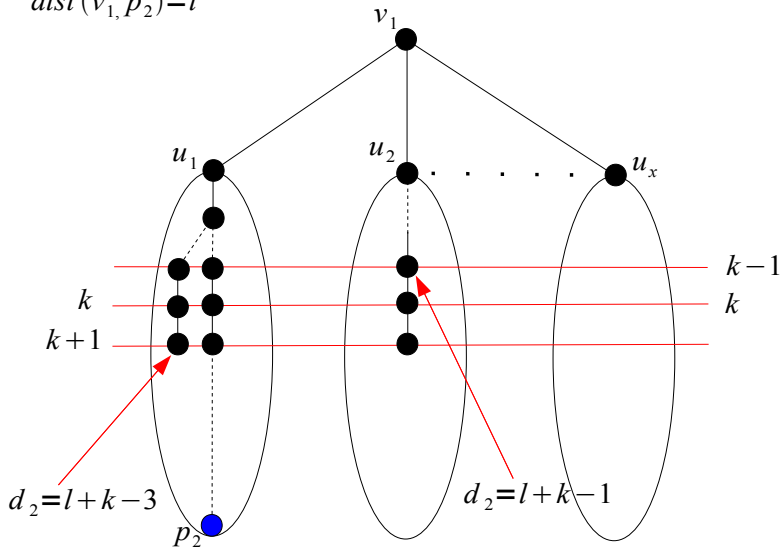
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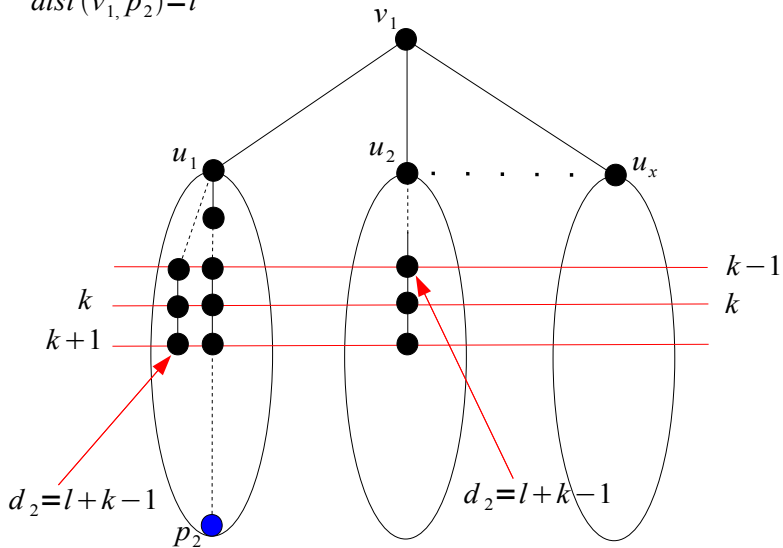
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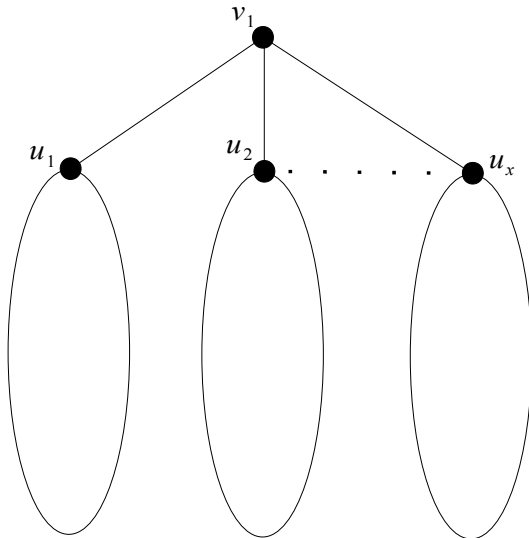


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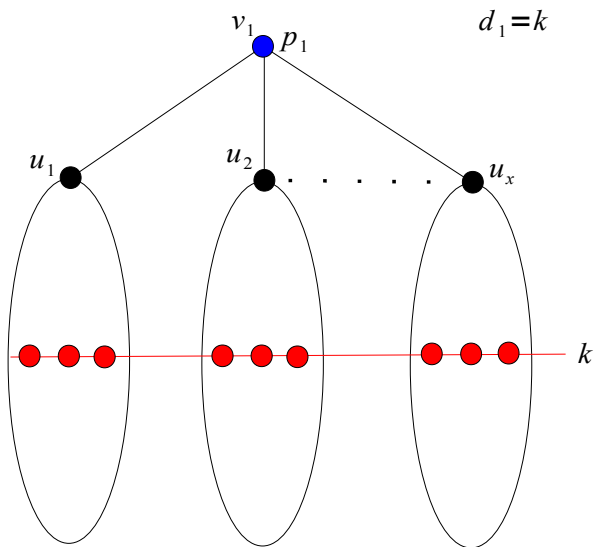
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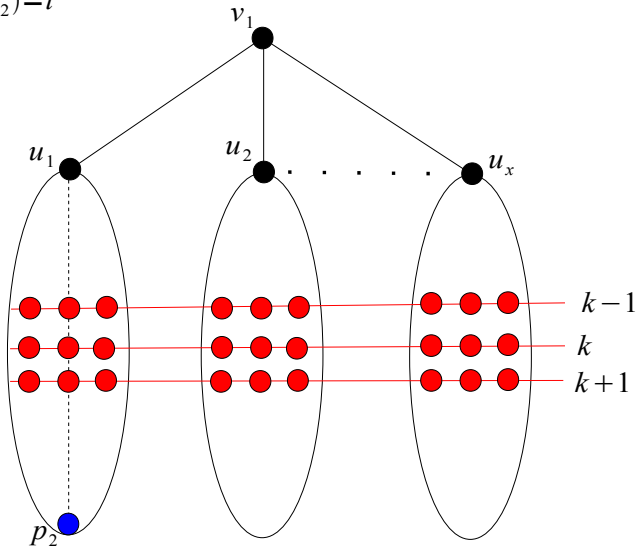


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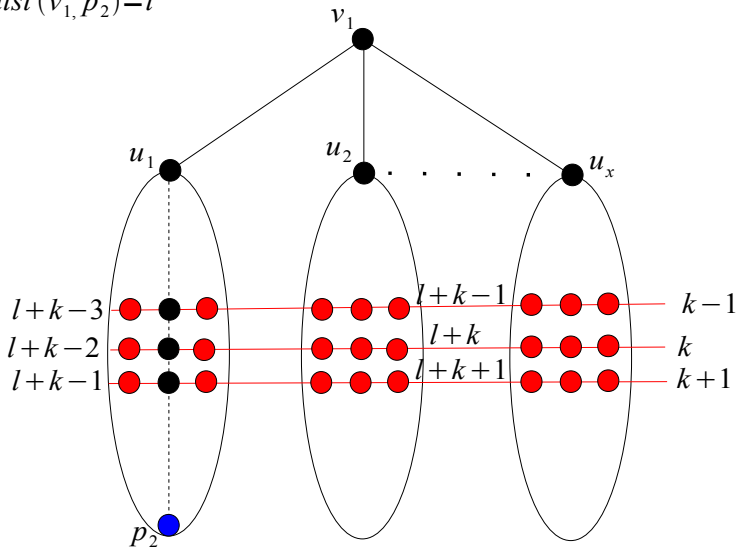
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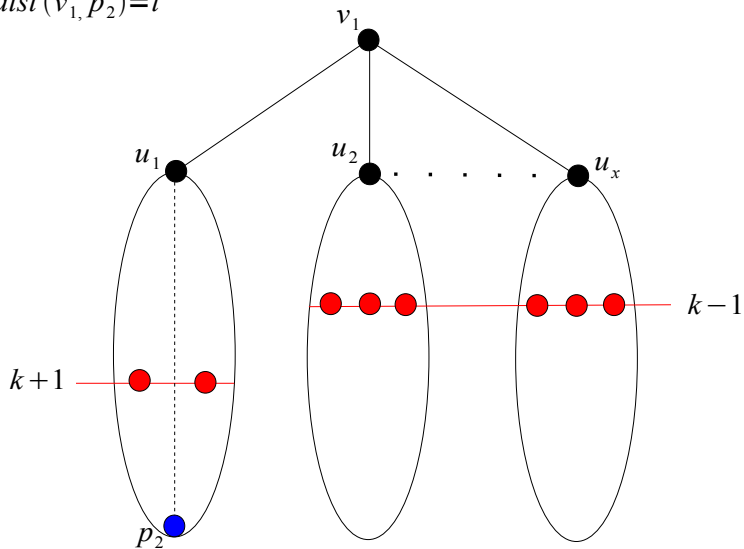
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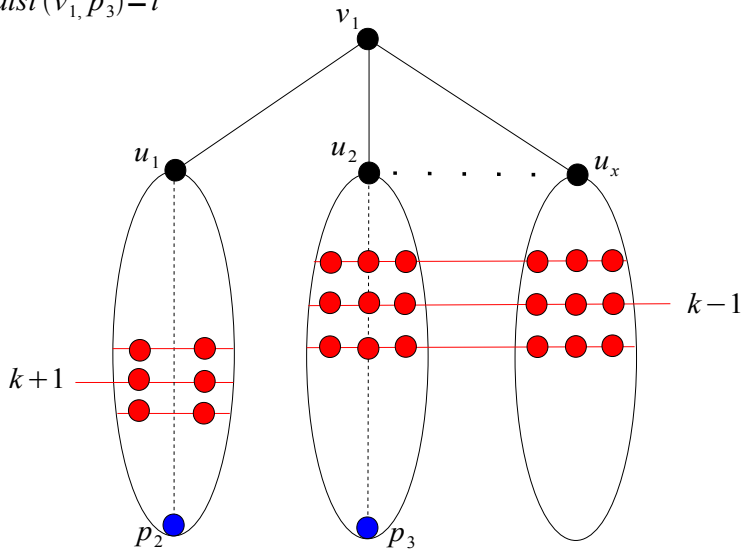
Algorithm Phase 3:

$$\text{dist}(v_1, p_2) = l$$



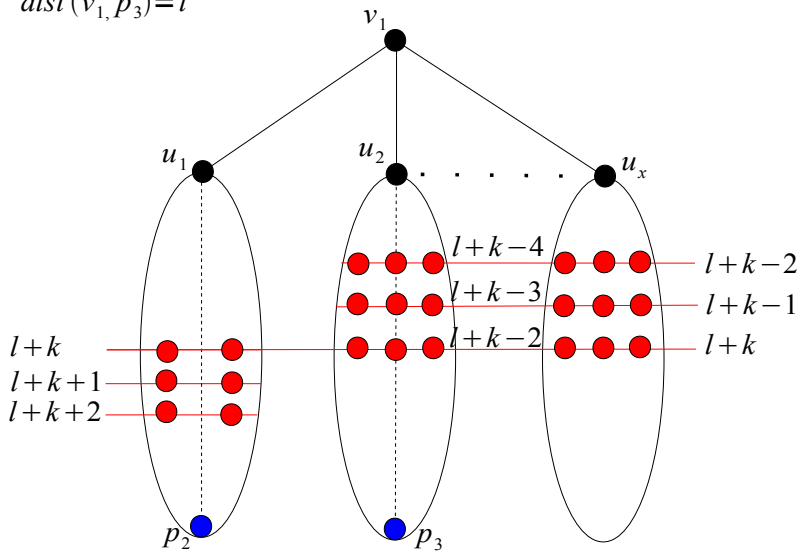
Algorithm Phase 3:

$$\text{dist}(v_1, p_3) = l$$



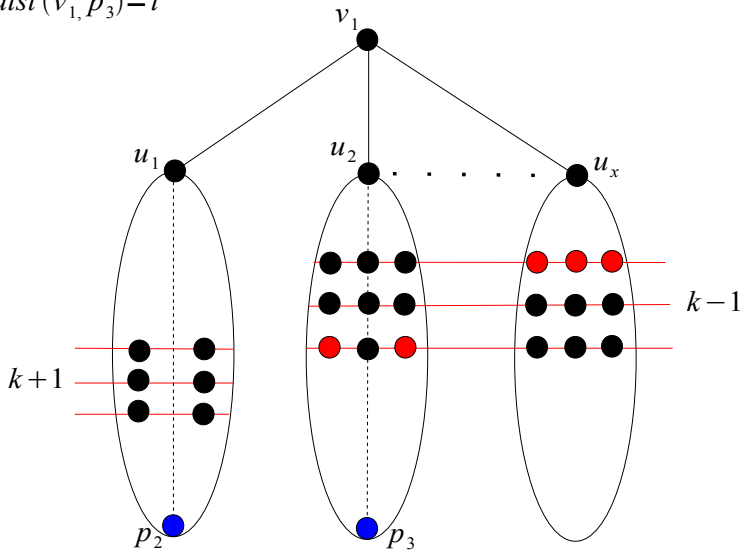
Algorithm Phase 3:

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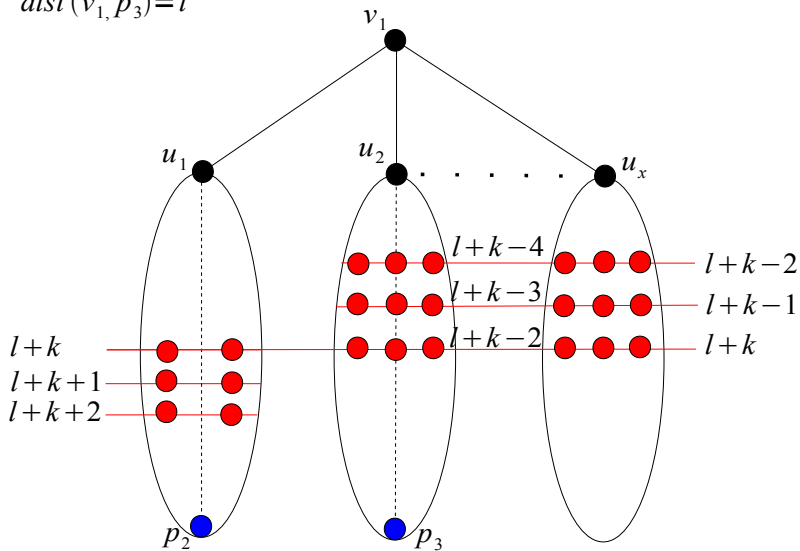
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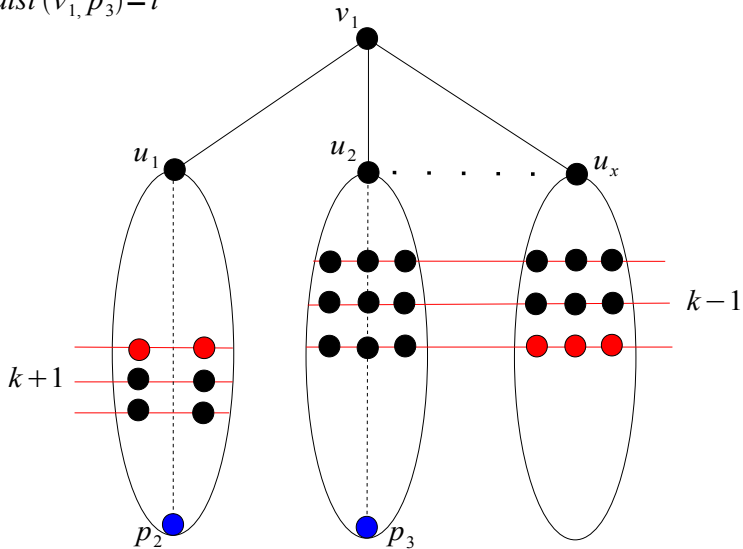
Algorithm Phase 3:

$$\text{dist}(v_1, p_3) = l$$

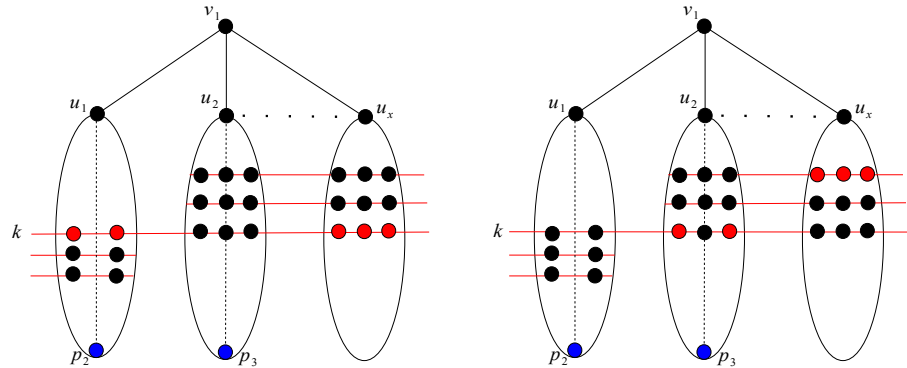


Algorithm Phase 3:

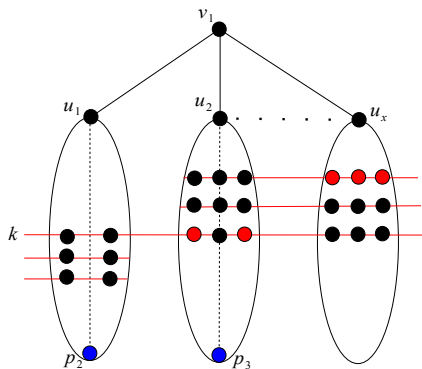
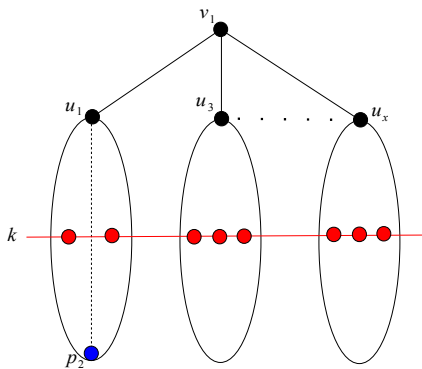
$$\text{dist}(v_1, p_3) = l$$



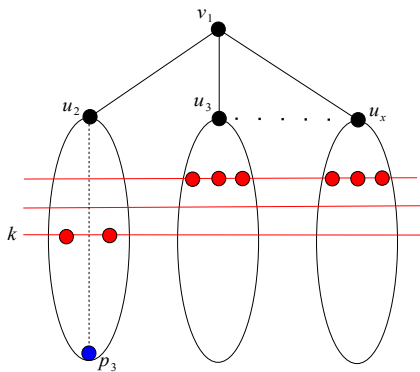
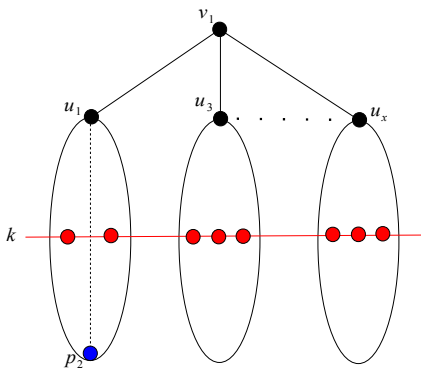
Algorithm Phase 3:



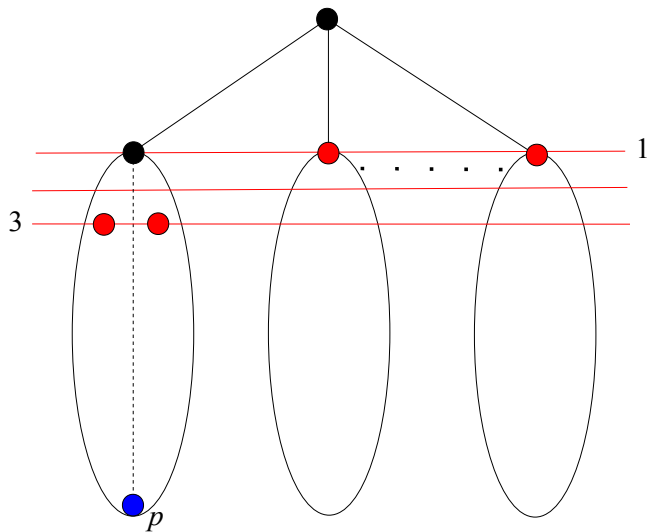
Algorithm Phase 3:



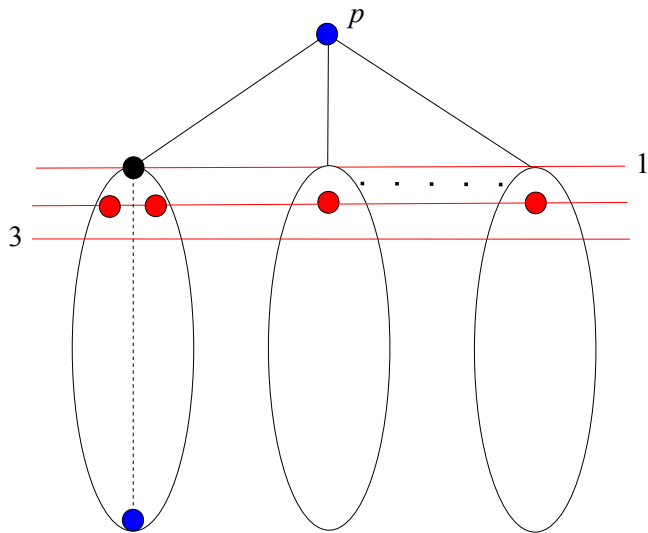
Algorithm Phase 3:



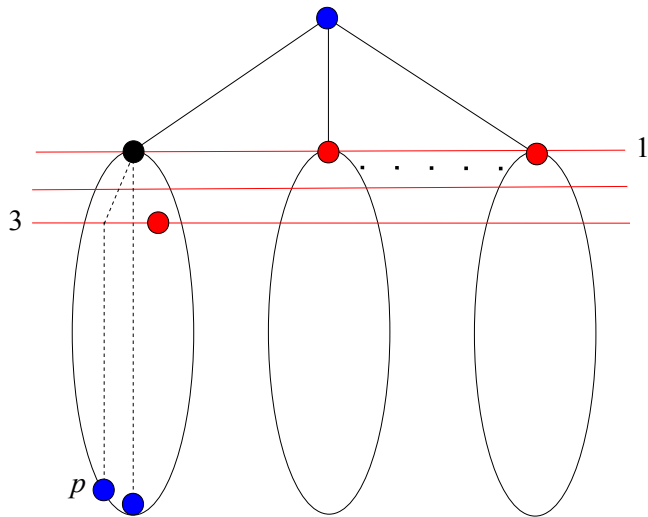
Algorithm Phase 3:



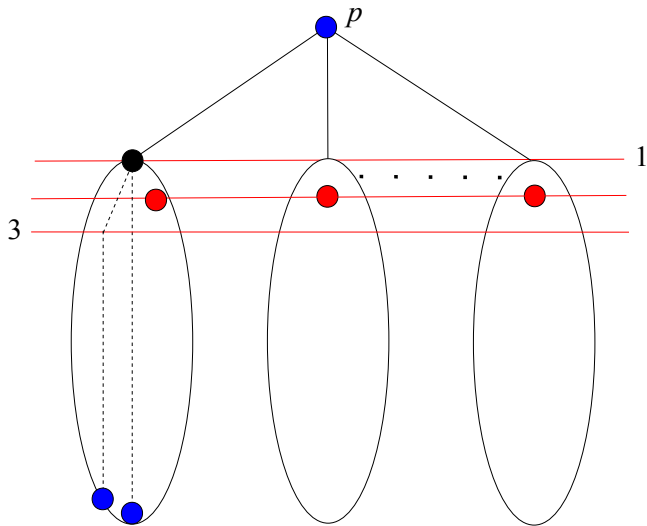
Algorithm Phase 3:



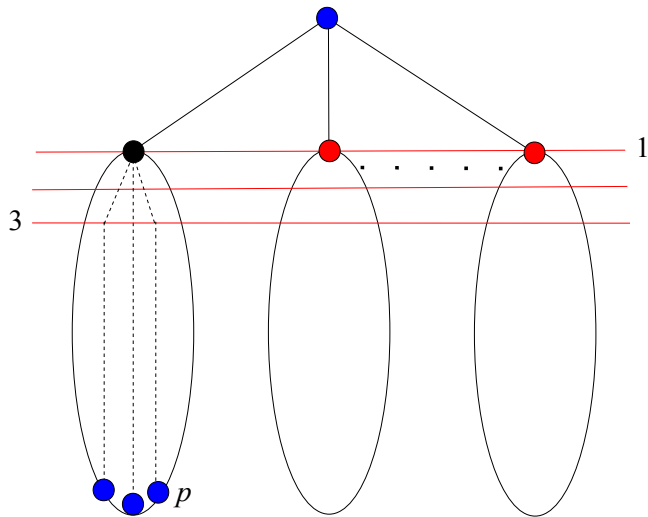
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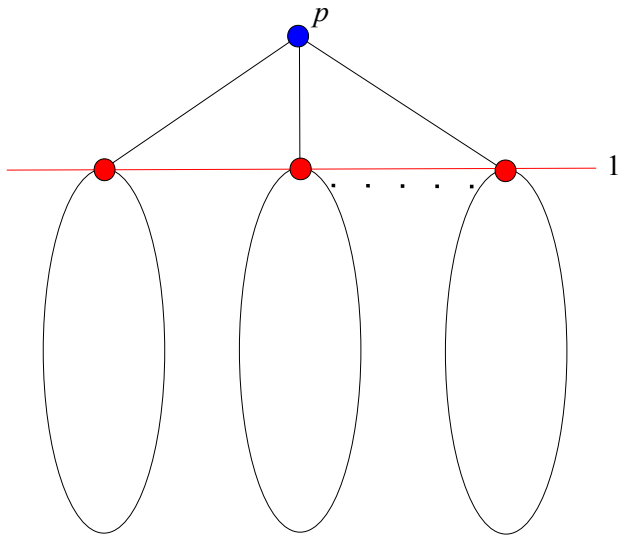


Algorithm Phase 3:

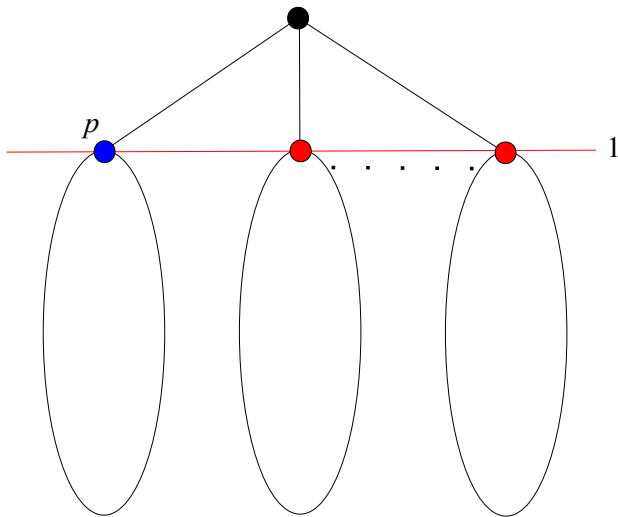


Algorithm Phase 4:

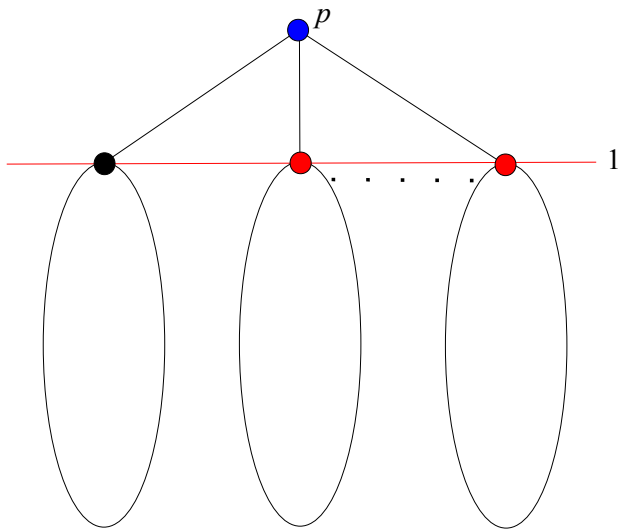
Algorithm Phase 4:



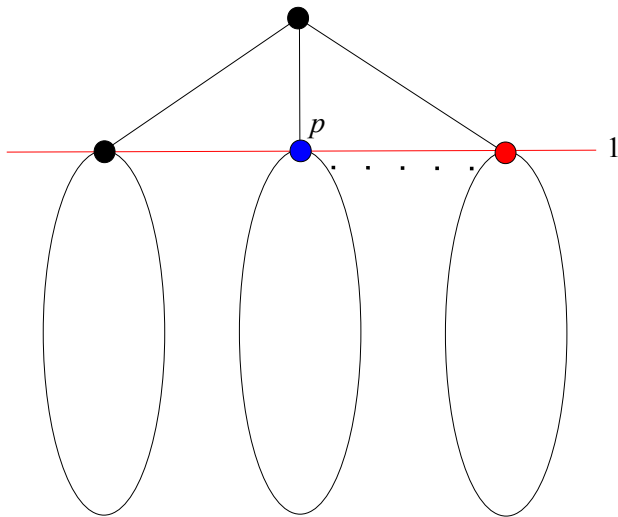
Algorithm Phase 4:



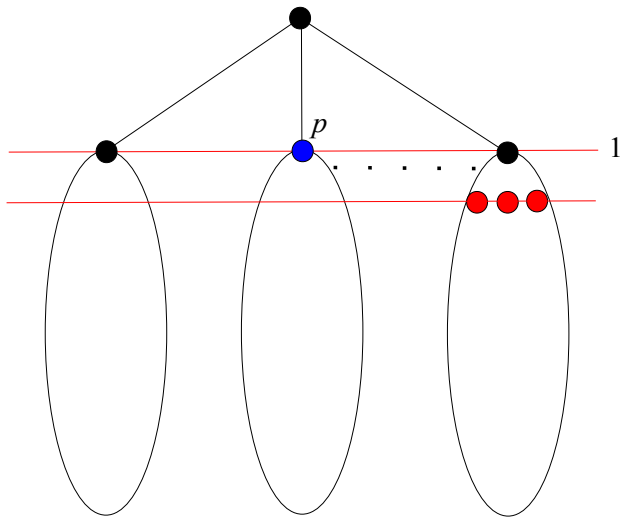
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Algorithm Upper Bound

Lemma

If $T \subseteq T'$ then $loc(T) \leq loc(T')$

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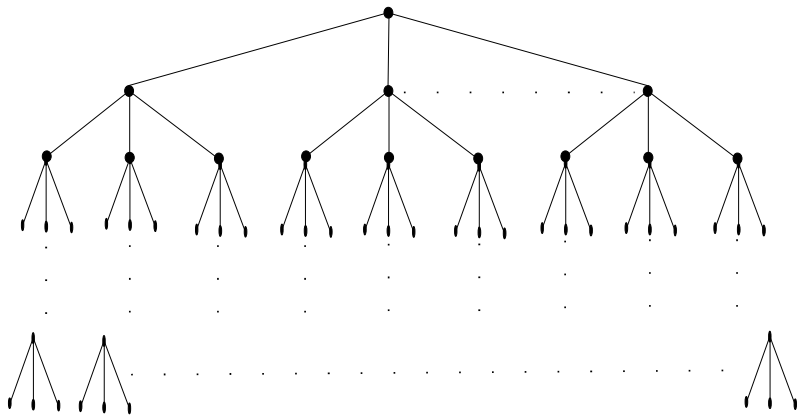
- Let T' be the smallest Δ -inary tree such that $T \subseteq T'$.

Lemma

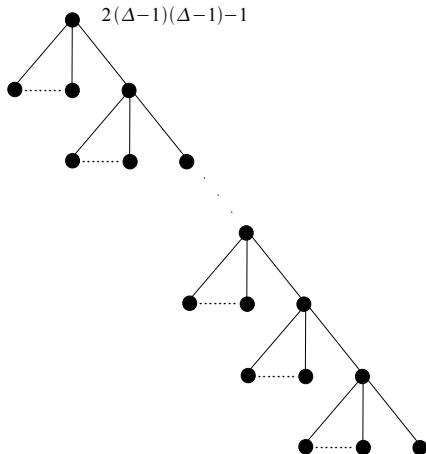
If $T \subseteq T'$ then $loc(T) \leq loc(T')$

- Let T' be the smallest Δ -inry tree such that $T \subseteq T'$.
- Determining $loc(T')$ gives an upper bound on $loc(T)$.

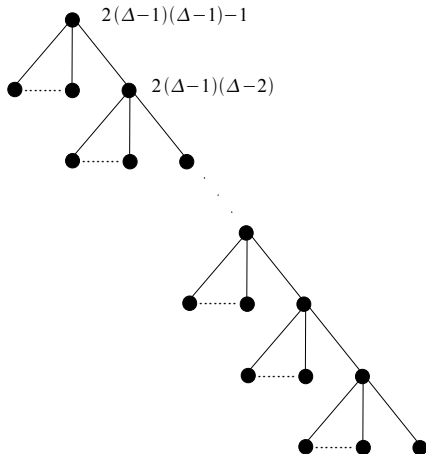
Algorithm Upper Bound (Diameter at least 6).



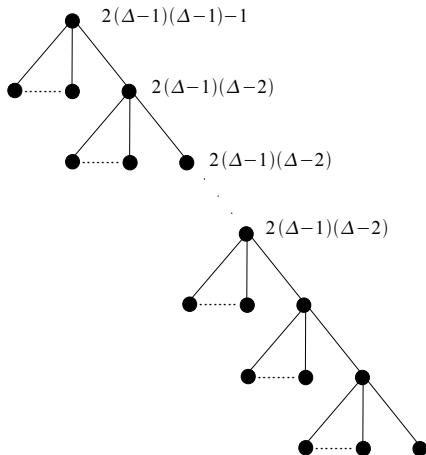
Algorithm Upper Bound (Diameter at least 6).



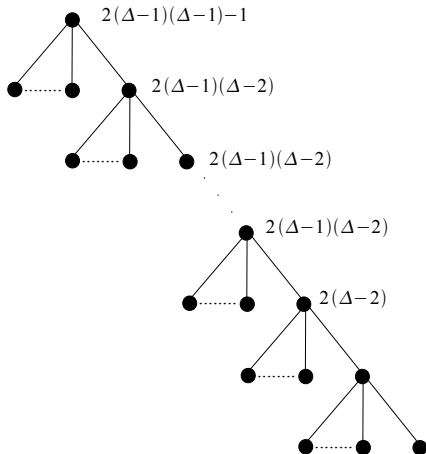
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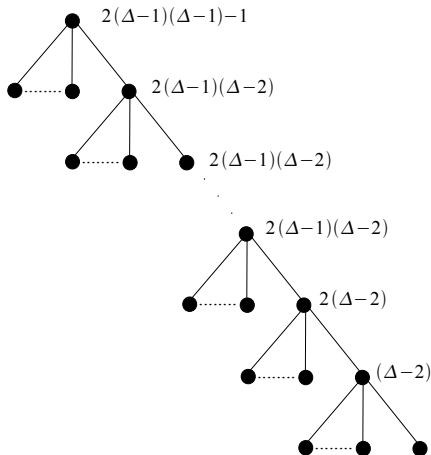
Algorithm Upper Bound (Diameter at least 6).



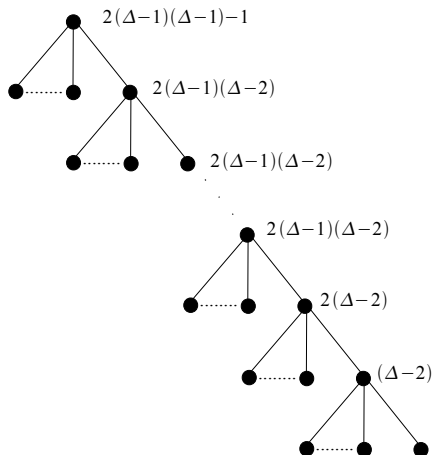
Algorithm Upper Bound (Diameter at least 6).



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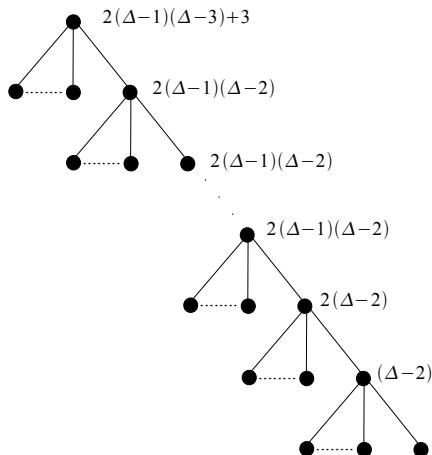


Algorithm Upper Bound (Diameter at least 6).



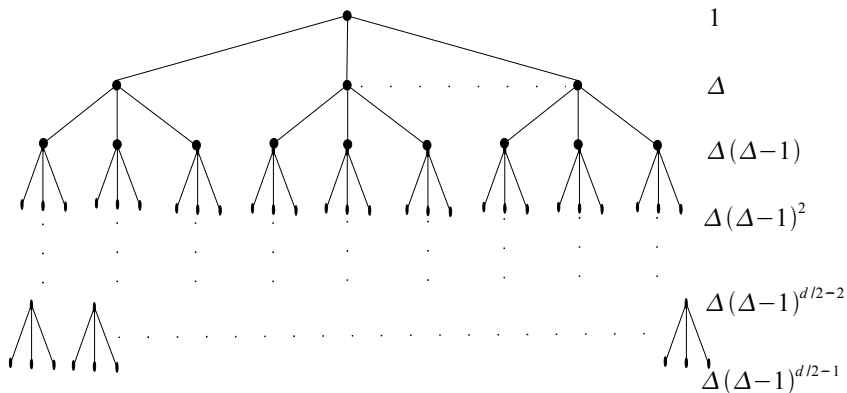
$$loc(T) \leq (2\Delta^2 - 6\Delta + 4) \lceil \frac{d}{2} \rceil - 4\Delta^2 + 17\Delta - 17$$

Algorithm Upper Bound (Diameter at least 6).



$$loc(T) \leq (2\Delta^2 - 6\Delta + 4) \lceil \frac{d}{2} \rceil - 4\Delta^2 + 13\Delta - 9$$

Compare with $n - 2$ For k -inary Tree.



$$\sum_{i=0}^{d/2-1} \Delta(\Delta-1)^i + 1$$

Compare with $n - 2$ For k -inary Tree.

$$f(\Delta, d) = (2\Delta^2 - 6\Delta + 4)\lceil \frac{d}{2} \rceil - 4\Delta^2 + 13\Delta - 9$$

- $f(\Delta, d) \leq d\Delta^2$

Compare with $n - 2$ For k -inary Tree.

$$f(\Delta, d) = (2\Delta^2 - 6\Delta + 4)\lceil \frac{d}{2} \rceil - 4\Delta^2 + 13\Delta - 9$$

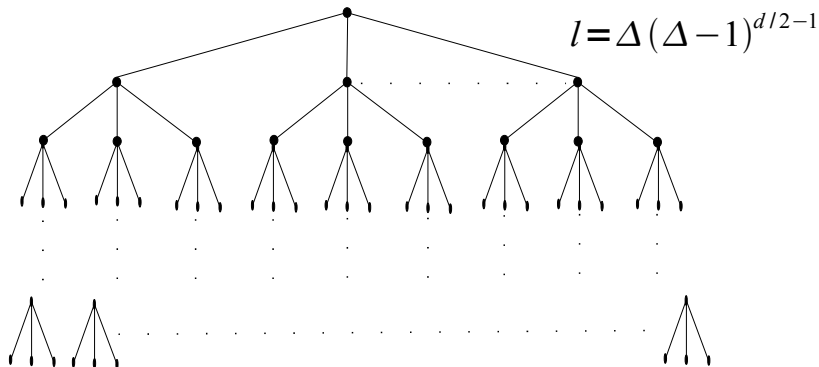
- $f(\Delta, d) \leq d\Delta^2$
- $d\Delta^2 < \sum_{i=0}^{\frac{d}{2}-1} (\Delta)(\Delta - 1)^i - 1$

Compare with $n - 2$ For k -inary Tree.

$$f(\Delta, d) = (2\Delta^2 - 6\Delta + 4)\lceil \frac{d}{2} \rceil - 4\Delta^2 + 13\Delta - 9$$

- $f(\Delta, d) \leq d\Delta^2$
- $d\Delta^2 < \sum_{i=0}^{\frac{d}{2}-1} (\Delta)(\Delta - 1)^i - 1$
- So $f(\Delta, d) \ll n - 2$ for Δ -inary tree.

Compare with $\ell - 1 \dots ?$



$$loc(T) \leq (2\Delta^2 - 6\Delta + 4) \left\lceil \frac{d}{2} \right\rceil - 4\Delta^2 + 13\Delta - 9$$

Compare with $\ell - 1 \dots ?$

Phase 1: Each ping we delete one leaf.

Compare with $\ell - 1$...?

Phase 1: Each ping we delete one leaf.

Phase 2: Every two pings we delete at least two leaves.

Compare with $\ell - 1 \dots ?$

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Phase 3: Each leaf takes at most two pings (except first/last).

Compare with $\ell - 1$...?

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Phase 4: Half of the leaves take at most two pings.

Compare with $\ell - 1$...?

Phase 1: Each ping we delete one leaf.

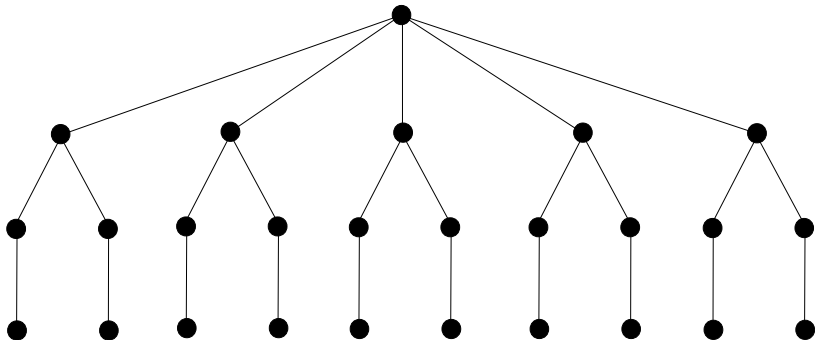
Phase 2: Every two pings we delete at least two leaves.

Phase 3: Each leaf takes at most two pings (except first/last).

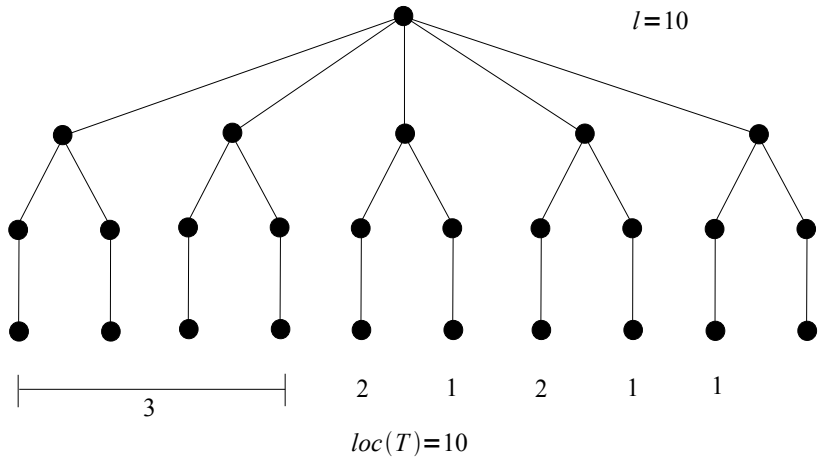
Phase 4: Half of the leaves take at most two pings.

$$\Rightarrow \text{loc}(T) < 2\ell.$$

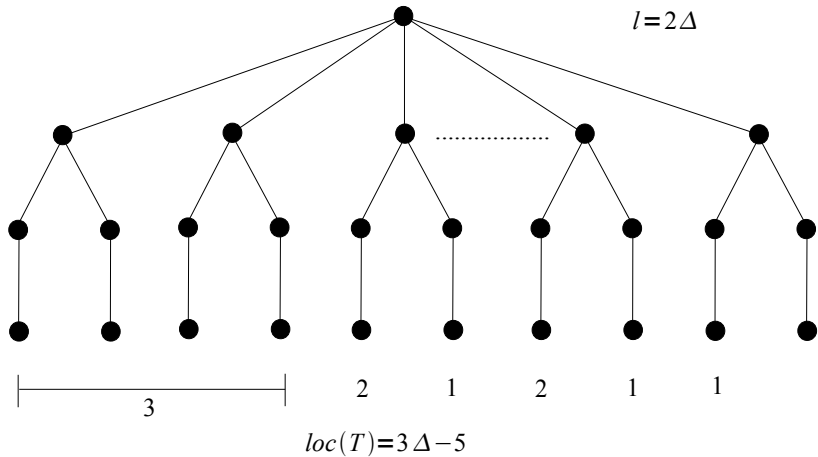
Counter Example



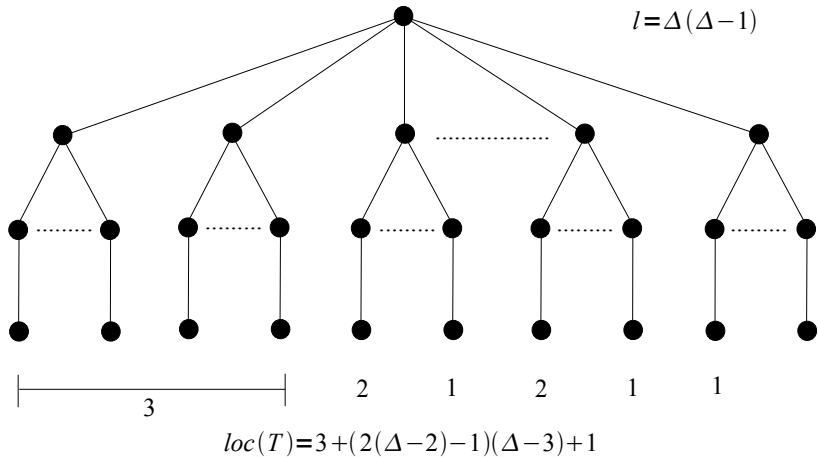
Counter Example



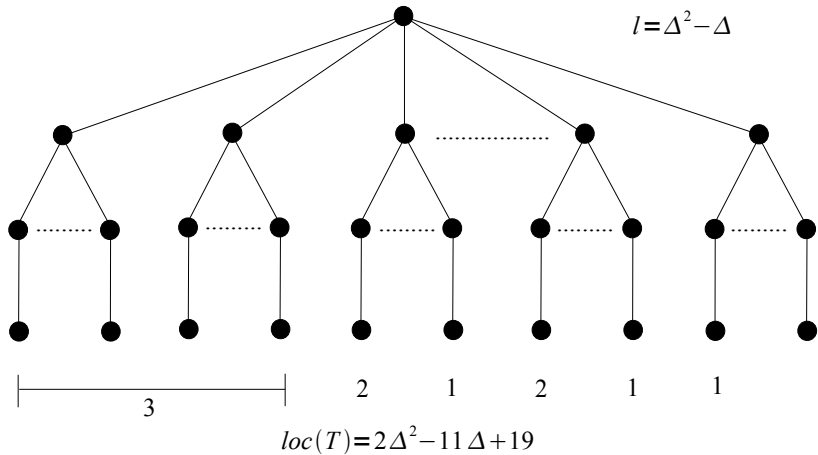
Counter Example



Counter Example



Counter Example



Question

Lower Bound as a Function of ℓ ?

Question

Lower Bound as a Function of ℓ ?

Question

Can we generalize this to larger classes?

Thank You

Thank You

Seager. **Locating a robber on a graph.** *Discrete Math.* 312 (2012), no. 22, 3265-3269.