On The Sum Necessary to Ensure that a Degree Sequence is Potentially H-Graphic.

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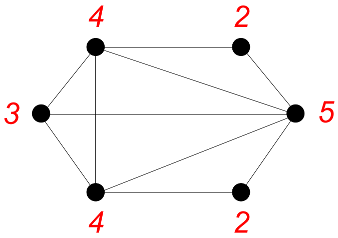

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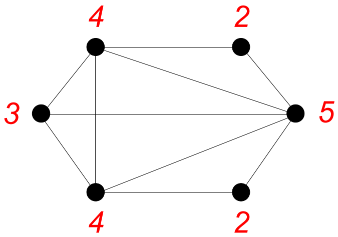

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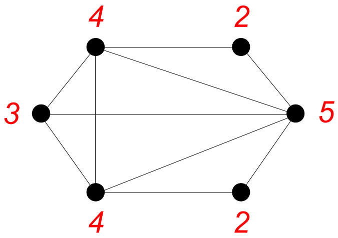

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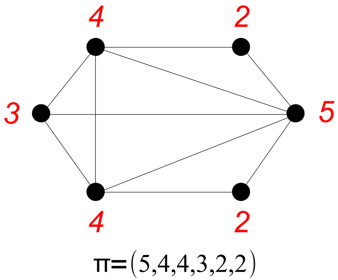
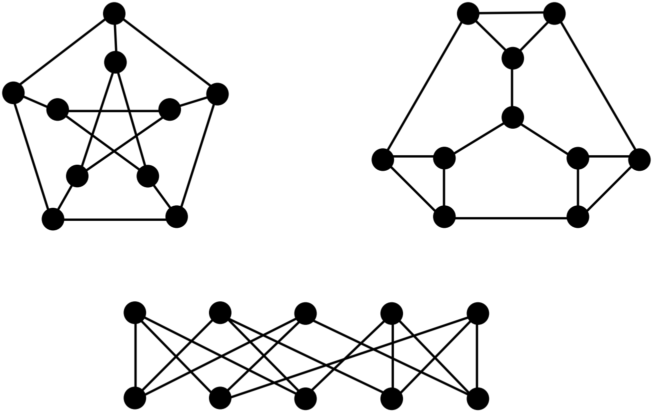


Such a graph G is said to **realize** π. We write $G(\pi) = G$.

Diversity of Realizations

It is easy to determine which sequences are graphic, but any given graphic sequence can have many distinct realizations.



Forcible/Potential

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Forcible Degree Sequence Problems

Consider the classic result by Dirac.

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Theorem (Dirac's Theorem 1952)

Let G be a graph of order $n \geq 3$. If $\delta(G) \geq \frac{n}{2}$ then G is hamiltonian. This result is sharp.

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Theorem (Dirac's Theorem 1952)

Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence with $n \geq 3$. If $d_n \geq \frac{n}{2}$, then π is forcibly hamiltonian.

Consider another example, let H be any arbitrary graph.

Problem (The Turán Problem)

Determine $ex(n, H)$, the maximum number of edges in an n-vertex graph that does not contain H.

To rephrase this as a degree sequence problem for $\pi = (d_1, \ldots, d_n)$ we will define,

$$\sigma(\pi) = \sum_{i=1}^{n} d_i,$$

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Noting that $\sigma(\pi(G)) = 2|E(G)|$. We can now restate the Turán problem.

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Determine $ex(n, H)$, the minimum integer such that every n-term graphic sequence π with

$$\sigma(\pi) \geq 2ex(H, n)$$

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History

If π is a graphic sequence, and some realization of π contains H as a subgraph, then we say that π is **potentially H-graphic.**

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Determine $\sigma(H, n)$, the minimum even integer such that any n-term graphic sequence with

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We refer to $\sigma(H, n)$ as the **potential number** of H.

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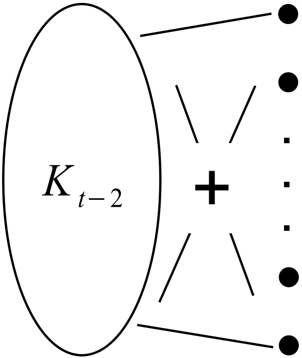

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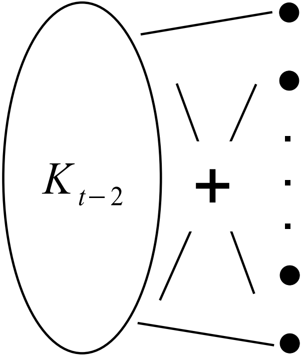

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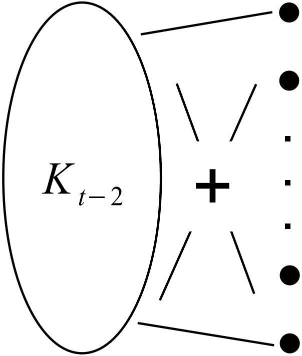

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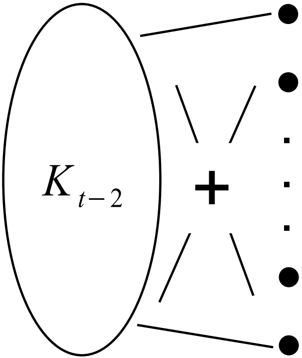

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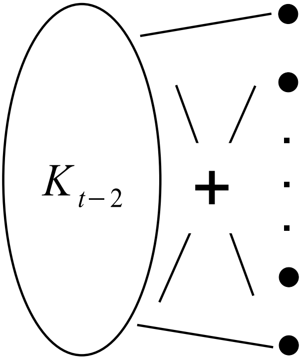

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$\pi = ((n-1)^{t-2}, (t-2)^{n-t+2})$ $\Rightarrow$ $\sigma(K_t, n) \geq (t-2)(2n-t+1)+2$

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In 1999 Li, Song, and Luo proved the conjecture for $t \geq 6$, and n sufficiently large.

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Building on a previous construction by Ferrara, and Schmitt (2009), we determine $\sigma(H, n)$ asymptotically for all H.

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Consider any $F_i \leq H$ with $\Delta(F_i) = \nabla_i(H)$. If F is any subgraph of H with order i, then

$$\Delta(F) \geq \Delta(F_i).$$

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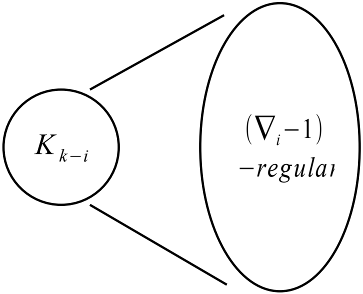

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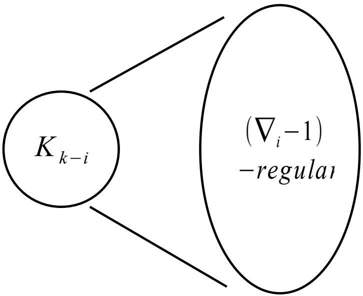

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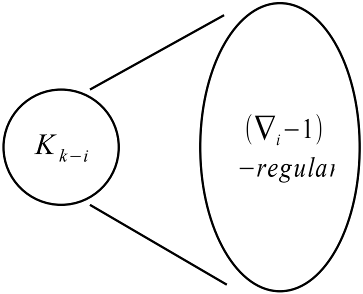

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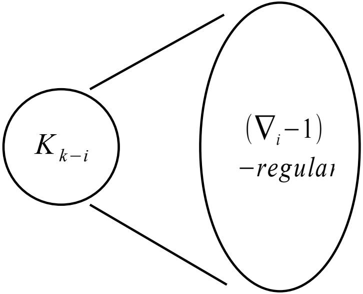

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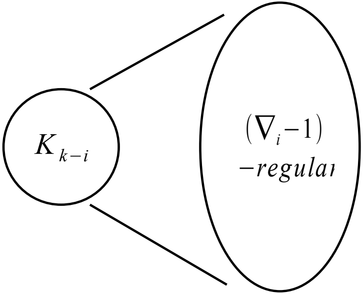

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$$\sigma(\pi_i) \approx (2(k-i) + \nabla_i - 1)n \quad \Rightarrow \quad \sigma(H, n) \geq \sigma(\pi_i) + 2$$

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Theorem (Ferrara, Moffatt, LeSaulnier, Wenger 2012+)

If H is a graph and n is a sufficiently large positive integer, then

$$\sigma(H, n) = \tilde{\sigma}(H)n + o(n).$$

Example: $K_{1,k-1}$

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$$\widetilde{\sigma}(K_{1,k-1}) = (2(k - k) + \nabla_k) = k - 1$$

Claim

$$\sigma(K_{1,k-1}, n) \approx (k - 1)n$$

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$$\widetilde{\sigma}_i(K_k) = (2(k-i) + (i-1) - 1) = 2k - i - 2$$

Claim

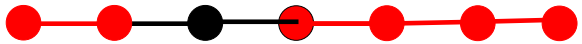
$$\sigma(K_k, n) \approx (2k-4)n$$

Example: P_7

$$\alpha(P_7) = 4$$

$$\nabla_5 = 1$$

$$\sigma_4 = (2(7-5) + 1 - 1) = 4$$



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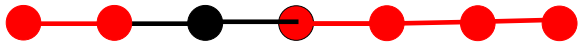

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Claim

$$\sigma(P_7, n) \approx 4n$$

Proof Sketch:

Let H be a graph of order k, and let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence with

$$\sigma(\pi) \geq \widetilde{\sigma}(H)n + \omega(n),$$

where $\omega(n) \xrightarrow{n} \infty$.

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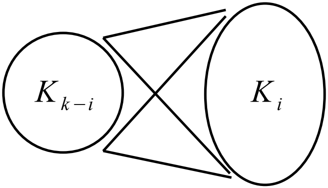

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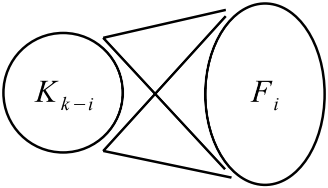

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There exists a function $f = f(\alpha(H), k)$ such that if

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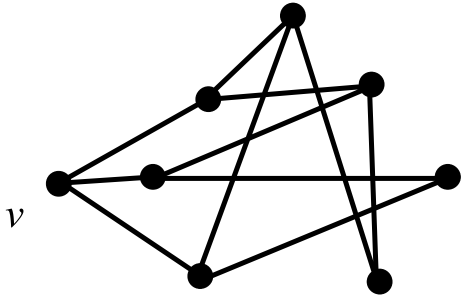
Consider any realization G of π, and choose a vertex $v \in G$.



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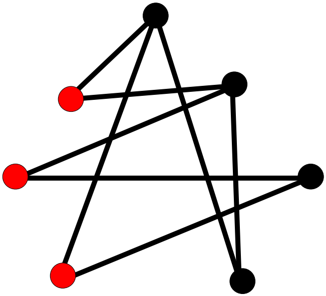

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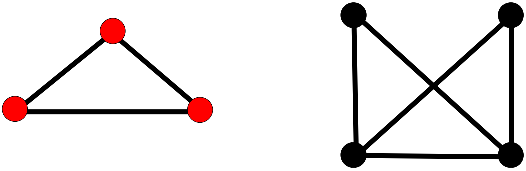

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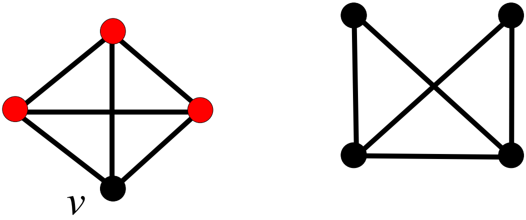

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1. A “dense" graph with a potentially H-graphic degree sequence.
2. A sufficiently long degree sequence π' of length n' with

$$\sigma(\pi') \geq \widetilde{\sigma}(H)n' \quad \text{and} \quad d_{n'} \geq k-\alpha.$$

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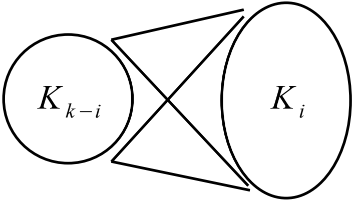
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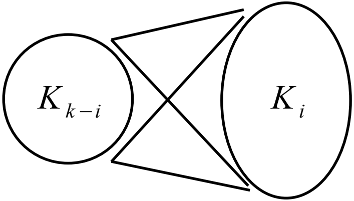
K_i



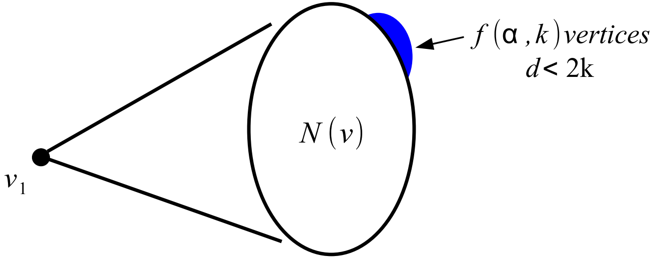
We can show that for some t, π' is degree sufficient for

$$K_t \vee \overline{K}_{k-t}.$$

If π' is not $(K_t \vee \overline{K}_{k-t})$-graphic, then by the Bounded Max Degree Theorem,

$$d_1 \geq n - f(\alpha, k).$$

The vertices not adjacent to v have degree at most $2k$ and n is sufficiently large, so we can delete them without changing the asymptotics of $\sigma(\pi)$.



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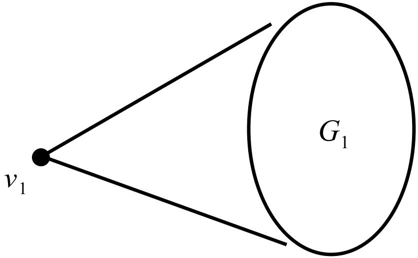
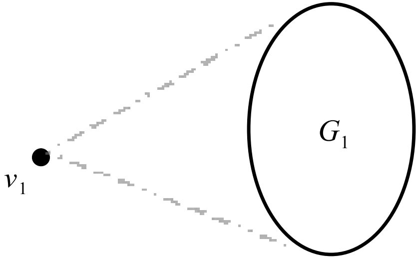
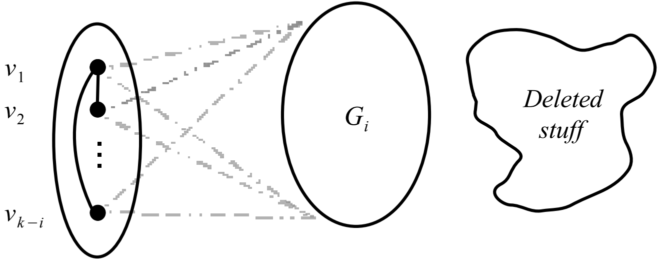


Since the vertices not adjacent to v have degree at most $2k$ and n is sufficiently large, we can delete them without changing the asymptotics of $\sigma(\pi)$.



Wash, Rinse, Repeat

We now repeat this process as many times as is possible and get



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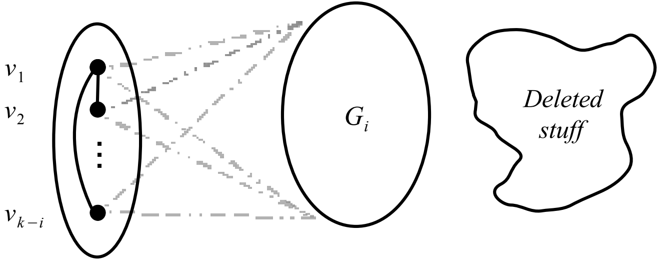

We can further show that:

1. G_i is sufficiently large,

Wash, Rinse, Repeat

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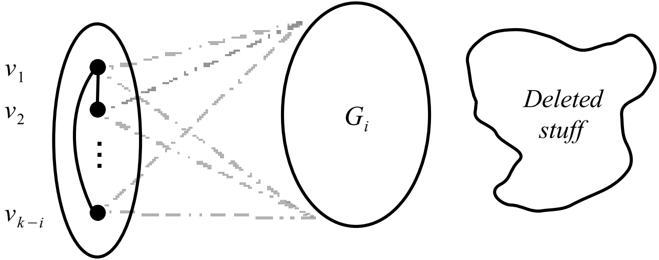
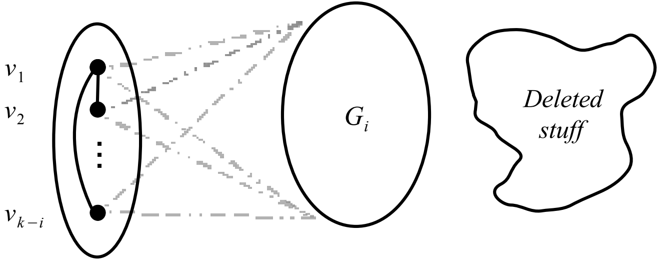


We can further show that:

1. G_i is sufficiently large,
2. $k - 1 \geq \Delta(G_i) \geq \delta(G_i) \geq 1$, and

We now repeat this process as many times as is possible and get



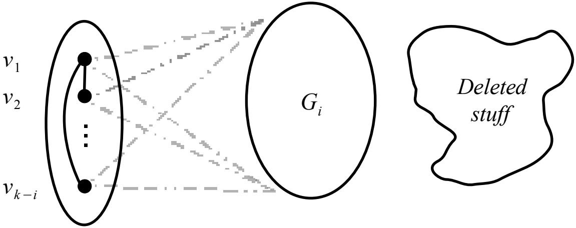
We can further show that:

1. G_i is sufficiently large,
2. $k - 1 \geq \Delta(G_i) \geq \delta(G_i) \geq 1$, and
3. G_i is degree sufficient for F_i.

And now, just build G_i

Lemma

Let π be an n-term graphic sequence with $d_1 \leq M$ and $d_n \geq 1$. If π is degree sufficient for H, and $n > 2M^2 + k$, then π is potentially H-graphic.



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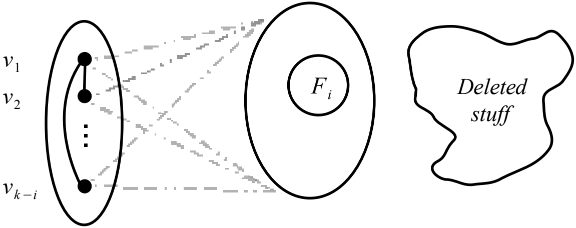

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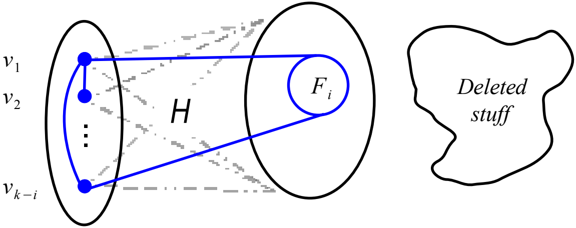

Conclusion

Conclusion

- Multigraphs...?

QUESTIONS?