

Rainbow Matchings of Size $\delta(G)$ in Properly Edge-Colored Graphs

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Abstract

A *rainbow matching* in an edge-colored graph is a matching in which all the edges have distinct colors. Wang asked if there is a function $f(\delta)$ such that a properly edge-colored graph G with minimum degree δ and order at least $f(\delta)$ must have a rainbow matching of size δ . We answer this question in the affirmative; $f(\delta) = 6.5\delta$ suffices. Furthermore, the proof provides a $O(\delta(G)|V(G)|^2)$ -time algorithm that generates such a matching.

Keywords: Rainbow matching, properly edge-colored graphs

1 Introduction

All graphs under consideration in this paper are simple, and we let $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degree of a graph G , respectively. A *rainbow subgraph* in an edge-colored graph is a subgraph in which all edges have distinct colors. Rainbow matchings are of particular interest given their connection to transversals of Latin squares: each Latin square can be converted to a properly edge-colored complete bipartite graph, and a transversal of the Latin square is a perfect rainbow matching in the graph. Ryser's conjecture [2] that every Latin square of odd order has a transversal can be seen as the beginning of the study of rainbow matchings. Stein [5] later conjectured that every Latin square of order n has a transversal of size $n - 1$; equivalently every properly edge-colored $K_{n,n}$ has a rainbow

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matching of size $n - 1$. The connection between Latin transversals and rainbow matchings in $K_{n,n}$ has inspired additional interest in the study of rainbow matchings in triangle-free graphs.

Several results have been attained for rainbow matchings in arbitrarily edge-colored graphs. The *color degree* of a vertex v in an edge-colored graph G , written $\hat{d}(v)$, is the number of different colors on edges incident to v . We let $\hat{\delta}(G)$ denote the minimum color degree among the vertices in G . Wang and Li [7] proved that every edge-colored graph G contains a rainbow matching of size at least $\left\lceil \frac{5\hat{\delta}(G)-3}{12} \right\rceil$, and conjectured that $\left\lceil \hat{\delta}(G)/2 \right\rceil$ could be guaranteed when $\hat{\delta}(G) \geq 4$. LeSaulnier et al. [4] then proved that every edge-colored graph G contains a rainbow matching of size $\left\lfloor \hat{\delta}(G)/2 \right\rfloor$. Finally, Kostochka and Yancey [3] proved the conjecture of Wang and Li in full, and also that triangle-free graphs have rainbow matchings of size $\left\lfloor \frac{2\hat{\delta}(G)}{3} \right\rfloor$.

Since the edge-colored graphs generated by Latin squares are properly edge-colored, it is of interest to consider rainbow matchings in properly edge-colored graphs. In this direction, LeSaulnier et al. proved that a properly edge-colored graph G satisfying $|V(G)| \neq \delta(G) + 2$ that is not K_4 has a rainbow matching of size $\lceil \delta(G)/2 \rceil$. Wang then asked if there is a function f such that a properly edge-colored graph G with minimum degree δ and order at least $f(\delta)$ must contain a rainbow matching of size δ [6]. As a first step towards answering this question, Wang showed that a graph G with order at least $\frac{8\delta}{5}$ has a rainbow matching of size $\left\lfloor \frac{3\delta(G)}{5} \right\rfloor$.

In this paper we answer Wang's question from [6] in the affirmative.

Theorem 1. *If G is a properly edge-colored graph satisfying $|V(G)| > \frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta}$, then G contains a rainbow matching of size $\delta(G)$.*

If G is triangle-free, a smaller order suffices.

Theorem 2. *If G is a triangle-free properly edge-colored graph satisfying $|V(G)| > \frac{49}{8}\delta - \frac{21}{2} + \frac{9}{2\delta}$, then G contains a rainbow matching of size $\delta(G)$.*

The proofs of Theorems 1 and 2 depend on the implementation of a greedy algorithm, a significantly different approach than those found in [3], [4], [6], and [7]. This algorithm generates a rainbow matching in a properly edge-colored graph G in $O(\delta(G)|V(G)|^2)$ -time.

Since there are $n \times n$ Latin squares with no transversals (see [1]) when n is even, it is clear that $f(\delta) > 2\delta$ when δ is even. Furthermore, since maximum matchings in $K_{\delta,n-\delta}$ have only δ edges (provided $n \geq 2\delta$), there is no function for the order of G depending on $\delta(G)$ that can guarantee a rainbow matching of size greater than δ .

2 Proof of the Main Results

Proof of Theorem 1. We proceed by induction on $\delta(G)$. The result is trivial if $\delta(G) = 1$. We assume that G is a graph with minimum degree δ and order greater than $\frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta}$.

Lemma 3. *If G satisfies $\Delta(G) > 3\delta - 3$, then G has a rainbow matching of size δ .*

Proof. Let v be a vertex of maximum degree in G . By induction, there is a rainbow matching M of size $\delta - 1$ in $G - v$. Since v is incident to at least $3\delta - 2$ edges with distinct colors, there is an edge incident to v that is not incident to any edge in M and also has a color that does not appear in M . Thus there is a rainbow matching of size δ in G . \square

Lemma 4. *If G has a color class containing at least $2\delta - 1$ edges, then G has a rainbow matching of size δ .*

Proof. Let C be a color class with at least $2\delta - 1$ edges. By induction, there is a rainbow matching M of size $\delta - 1$ in $G - C$. There are $2\delta - 2$ vertices covered by the edges in M , thus one of the edges in C has no endpoint covered by M , and the matching can be extended. \square

The proof of Theorem 1 relies on the implementation of a greedy algorithm. We begin by preprocessing the graph so that each edge is incident to at least one vertex with degree δ . To achieve this, we order the edges in G and process them in order. If both endpoints of an edge have degree greater than δ when it is processed, delete that edge. In the resulting graph, every edge is incident to a vertex with degree δ . Furthermore, by Lemma 3 we may assume that $\Delta(G) \leq 3\delta - 3$; thus the degree sum of the endpoints of any edge is bounded above by $4\delta - 3$. After preprocessing, we begin the greedy algorithm.

In the i th step of the algorithm, a smallest color class is chosen (without loss of generality, color i), and then an edge e_i of color i is chosen such that the degree sum of the endpoints is minimum. All the remaining edges of color i and all edges incident with an endpoint of e_i are deleted. The algorithm terminates when there are no edges in the graph.

We assume that the algorithm fails to produce a matching of size δ in G ; suppose that the rainbow matching M generated by the algorithm has size k . We let R denote the set of vertices that are not covered by M .

Let c_i denote the size of the smallest color class at step i . Since at most two edges of color $i + 1$ are deleted in step i (one at each endpoint of e_i), we observe that $c_{i+1} + 2 \geq c_i$. Otherwise, at step i color class $i + 1$ has fewer edges. Let step h be the last step in the algorithm in which a color class that does not appear in M is completely removed from G .

It then follows that $c_h \leq 2$, and in general $c_i \leq 2(h - i + 1)$ for $i \in [h]$. Let f_i denote the number of edges of color i deleted in step i with both endpoints in R . Since $f_i < c_i$, we have $f_i \leq 2(h - i) + 1$ for $i \in [h]$. Note that after step h , there are exactly $k - h$ colors remaining in G . By Lemma 4, color classes contain at most $2\delta - 2$ edges, and therefore the last $k - h$ steps remove at most $(k - h)(2\delta - 2)$ edges. Furthermore, for $i > h$, the degree sum of the endpoints of e_i is at most $2(\delta - 1)$.

For $i \in [h]$, let x_i and y_i be the endpoints of e_i , and let $d_i(v)$ denote the degree of a vertex v at the beginning of step i . Let $\mu_i = \max\{0, d_i(x_i) + d_i(y_i) - 2\delta\}$; note that $2\delta \leq 2\delta + \mu_i \leq 4\delta - 3$. Thus, at step i , at most $2\delta + \mu_i + f_i - 1$ edges are removed from the graph. Since the algorithm removes every edge from the graph, we conclude that

$$|E(G)| \leq (k - h)(2\delta - 2) + \sum_{i=1}^h (2\delta + \mu_i + f_i - 1). \quad (1)$$

We now compute a lower bound for the number of edges in G . Since the degree sum of the endpoints of e_i in G is at least $2\delta + \mu_i$, we immediately obtain the following inequality:

$$\frac{n\delta + \sum_{i \in [h]} \mu_i}{2} \leq |E(G)|.$$

If $f_i > 0$ and $\mu_i > 0$, then there is an edge with color i having both endpoints in R . Since this edge was not chosen in step i by the algorithm, the degree sum of its endpoints is at least $2\delta + \mu_i$, and one of its endpoints has degree at least $\delta + \mu_i$. For each value of i satisfying $f_i > 0$, we wish to choose a representative vertex in R with degree at least $\delta + \mu_i$. Since there are f_i edges with color i with both endpoints in R , there are f_i possible representatives for color i . Since a vertex in R with high degree may be the representative for multiple colors, we wish to select the largest system of distinct representatives.

Suppose that the largest system of distinct representatives has size t , and let T be the set of indices of the colors that have representatives. For each color $i \in T$ there is a distinct vertex in R with degree at least $\delta + \mu_i$. Thus we may increase the edge count of G as follows:

$$\frac{n\delta + \sum_{i \in [h]} \mu_i + \sum_{i \in T} \mu_i}{2} \leq |E(G)|. \quad (2)$$

We let $\{f_i^\downarrow\}$ denote the sequence $\{f_i\}_{i \in [h]}$ sorted in nonincreasing order. Since $f_i \leq 2(h - i) + 1$, we conclude that $f_i^\downarrow \leq 2(h - i) + 1$. Because there is no system of distinct representatives of size $t + 1$, the sequence $\{f_i^\downarrow\}$ cannot majorize the sequence $\{t + 1, t, t - 1, \dots, 1\}$. Hence there is a smallest value $p \in [t + 1]$ such that $f_p^\downarrow \leq t + 1 - p$. Therefore, the maximum value

of $\sum_{i=1}^h f_i^\downarrow$ is bounded by the sum of the sequence $\{2h-1, 2h-3, \dots, 2(h-p)+3, t+1-p, \dots, t+1-p\}$. Summing we attain

$$\sum_{i \in [h]} f_i \leq (p-1)(2h-p+1) + (h-p+1)(t+1-p).$$

Over p , this value is maximized when $p = t+1$, yielding $\sum_{i \in [h]} f_i \leq t(2h-t)$. Since $h \leq \delta-1$, we then have $\sum_{i \in [h]} f_i \leq 2(\delta-1)t - t^2$.

We now combine bounds (1) and (2):

$$\frac{n\delta + \sum_{i \in [h]} \mu_i + \sum_{i \in T} \mu_i}{2} \leq (k-h)(2\delta-2) + \sum_{i=1}^h (2\delta + \mu_i + f_i - 1).$$

Hence, since $k \leq \delta-1$,

$$\begin{aligned} \frac{n\delta}{2} &\leq (2\delta-1)(\delta-1) + \frac{1}{2} \sum_{[h] \setminus T} \mu_i + \sum_{i \in [h]} f_i \\ &\leq (2\delta-1)(\delta-1) + (\delta-1-t)(\delta-3/2) + 2(\delta-1)t - t^2 \\ &\leq 3\delta^2 - \frac{11}{2}\delta + \frac{5}{2} + \left(\delta - \frac{1}{2}\right)t - t^2. \end{aligned}$$

This bound is maximized when $t = (\delta - \frac{1}{2})/2$. Thus

$$n \leq \frac{13}{2}\delta - \frac{23}{2} + \frac{41}{8\delta},$$

contradicting our choice for the order of G . □

Sketch of Proof of Theorem 2. When G is triangle-free, Lemma 3 can be improved. In particular, $\Delta(G) \leq 2\delta - 2$ since there is at most one edge joining a vertex of maximum degree to each edge in a matching of size $\delta - 1$. Since $\Delta(G)$ is used to bound the value of μ_i in the proof of Theorem 1, the same argument yields the following inequality:

$$\begin{aligned} \frac{n\delta}{2} &\leq (2\delta-1)(\delta-1) + \frac{1}{2} \sum_{[h] \setminus T} \mu_i + \sum_{i \in [h]} f_i \\ &\leq (2\delta-1)(\delta-1) + \frac{1}{2}(\delta-1-t)(\delta-2) + 2(\delta-1)t - t^2 \\ &\leq \frac{5}{2}\delta^2 - \frac{9}{2}\delta + 2 + \left(\frac{3}{2}\delta - 1\right)t - t^2. \end{aligned}$$

This upper bound is maximized when $t = (\frac{3}{2}\delta - 1)/2$, yielding

$$n \leq \frac{49}{8}\delta - \frac{21}{2} + \frac{9}{2\delta}.$$

□

3 Conclusion

The proof of Theorem 1 provides the framework of a $O(\delta(G)|V(G)|^2)$ -time algorithm that generates a rainbow matching of size $\delta(G)$ in a properly edge-colored graph G . Given such a G , we create a sequence of graphs $\{G_i\}$ as follows, letting $G = G_0$, $\delta = \delta(G)$, and $n = |V(G)|$. First, determine $\delta(G_i)$, $\Delta(G_i)$, and the maximum size of a color class in G_i ; this process takes $O(n^2)$ -time. If $\Delta(G_i) \leq 3\delta(G_i) - 3$ and the maximum color class has at most $2\delta(G_i) - 2$ edges, then terminate the sequence and set $G_i = G'$. If $\Delta(G_i) > 3\delta(G_i) - 3$, then delete a vertex v of maximum degree and then process the edges of $G_i - v$, iteratively deleting those with two endpoints of degree at least $\delta(G_i)$; the resulting graph is G_{i+1} . If $\Delta(G_i) \leq 3\delta(G_i) - 3$ but a maximum color class C has at least $2\delta(G_i) - 1$ edges, then delete C and then process the edges of $G_i - C$, iteratively deleting those with two endpoints of degree at least $\delta(G_i)$; the resulting graph is G_{i+1} . Note that $\delta(G_{i+1}) = \delta(G_i) - 1$. If this process generates G_δ , we set $G' = G_\delta$ and terminate. Generating the sequence $\{G_i\}$ consists of at most δ steps, each taking $O(n^2)$ -time.

Given that $G' = G_i$, the algorithm from the proof of Theorem 1 takes $O(\delta n^2)$ -time to generate a matching of size $\delta - i$ in G' . The preprocessing step and the process of determining a smallest color class and choosing an edge in that class whose endpoints have minimum degree sum both take $O(n^2)$ -time. This process is repeated at most δ times.

A matching of size $\delta - (i + 1)$ in G_{i+1} is easily extended in G_i to a matching of size $\delta - i$ using the vertex of maximum degree or maximum color class. The process of extending the matching takes $O(\delta)$ -time. Thus the total run-time of the algorithm generating the rainbow matching of size δ in G is $O(\delta n^2)$.

It is worth noting that the analysis of the greedy algorithm used in the proof of Theorem 1 could be improved. In particular, the bound $c_{i+1} \geq c_i - 2$ is sharp only if at step i there are an equal number of edges of color i and $i + 1$ and both endpoints of e_i are incident to edges with color $i + 1$. However, since one of the endpoints of e_i has degree at most δ , at most $\delta - 1$ color classes can lose two edges in step i . Since the maximum size of a color class in G is at most $2\delta - 2$, if G has order at least 6δ , then there are at least $3\delta/2$ color classes. Thus, for small values of i , the bound $c_i \leq 2(k - i + 1)$ can likely be improved. However, we doubt that such analysis of this algorithm can be improved to yield a bound on $|V(G)|$ better than 6δ .

References

- [1] R. A. Brualdi and H. J. Ryser, *Combinatorial Matrix Theory*, Cambridge University Press, Cambridge, UK, 1991.
- [2] H. J. Ryser, Neuere Probleme der Kombinatorik, in “Vorträge über Kombinatorik Oberwolfach”. Mathematisches Forschungsinstitut Oberwolfach, July 1967, 24-29.
- [3] A. Kostochka and M. Yancey, Large Rainbow Matchings in Edge-Colored Graphs. In Preparation.
- [4] T. D. LeSaulnier, C. Stocker, P. S. Wenger, and D. B. West, Rainbow matching in edge-colored graphs. *Electron. J. Combin.* 17 (2010), Note #N26.
- [5] S. K. Stein, Transversals of Latin squares and their generalizations. *Pacific J. Math.* 59 (1975), no. 2, 567-575.
- [6] G. Wang, Rainbow matchings in properly edge colored graphs. *Electron. J. Combin.* 18 (2011), Paper #P162.
- [7] G. Wang and H. Li, Heterochromatic matchings in edge-colored graphs. *Electron. J. Combin.* 15 (2008), Paper #R138.