On the Sum Necessary to Ensure that a Degree Sequence is Potentially H-Graphic

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Abstract

A sequence of nonnegative integers $\pi = (d_1, d_2, ..., d_n)$ is *graphic* if there is a (simple) graph G with degree sequence π. In this case, G is said to *realize* or be a *realization of* π. Degree sequence results in the literature generally fall into two classes: *forcible* problems, in which all realizations of a graphic sequence must have a given property, and *potential* problems, in which at least one realization of π must have a given property.

Given a graph H, a graphic sequence π is *potentially H-graphic* if there is some realization of π that contains H as a subgraph. In 1991, Erdős, Jacobson and Lehel posed the following question:

Determine the minimum even integer $\sigma(H, n)$ such that every n-term graphic sequence with sum at least $\sigma(H, n)$ is potentially H-graphic.

As the sum of the terms of π is twice the number of edges in any realization of π, the Erdős-Jacobson-Lehel problem can be viewed as a potential degree sequence relaxation of the (forcible) Turán problem, wherein one wishes to determine the maximum number of edges in a graph that contains no copy of H.

While the exact value of $\sigma(H, n)$ has been determined for a number of specific classes of graphs (including cliques, cycles, complete bigraphs and others), very little is known about the parameter for arbitrary H. In this paper, we determine $\sigma(H, n)$ asymptotically for all H, thereby providing an Erdős-Stone-Simonovits-type theorem for the Erdős-Jacobson-Lehel problem.

Keywords: Degree sequence, potentially H-graphic sequence

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1 Introduction

A sequence of nonnegative integers $\pi = (d_1, d_2, ..., d_n)$ is *graphic* if there is a (simple) graph G of order n having degree sequence π. In this case, G is said to *realize* or be a *realization of* π, and we will write $\pi = \pi(G)$. If a sequence π consists of the terms $d_1, \ldots, d_t$ having multiplicities $\mu_1, \ldots, \mu_t$, we may write $\pi = ({d_1}^{\mu_1}, \ldots, {d_t}^{\mu_t})$. Unless otherwise noted, throughout this paper all sequences are nonincreasing. Additionally, we let $\sigma(\pi)$ denote the sum of the terms of π.

The study of graphic sequences dates to the 1950s, and includes the characterization of graphic sequences by Havel [18] and Hakimi [16] and an independent characterization by Erdős and Gallai [9]. Subsequent research sought to describe those graphic sequences that are realized by graphs with certain desired properties. Such problems can be broadly classified into two types, first described as "forcible" problems and "potential" problems by A.R. Rao in [28]. In a forcible degree sequence problem, a specified graph property must exist in every realization of the degree sequence π, while in a potential degree sequence problem, the desired property must be found in at least one realization of π.

Results on forcible degree sequences are often stated as traditional problems in structural or extremal graph theory, where a necessary and/or sufficient condition is given in terms of the degrees of the vertices (or equivalently the number of edges) of a given graph (e.g. Dirac's Theorem on hamiltonian graphs or the number of edges in a maximal planar graph). Two older, but exceptionally thorough surveys on forcible and potential problems are due to Hakimi and Schmeichel [17] and S.B. Rao [29].

A number of degree sequence analogues to classical problems in extremal graph theory appear throughout the literature, including potentially graphic sequence variants of Hadwiger's Conjecture [**?**, 30], the Sauer-Spencer graph packing theorem [1], and the Erdős-Sós Conjecture [25].

Pertinent to our work here is the *Turán Problem*, one of the most well-established and central problems in extremal graph theory.

Problem 1 (The Turán Problem)**.** *Let H be a graph and n be a positive integer. Determine the minimum integer $ex(H, n)$ such that every graph of order n with at least $ex(H, n) + 1$ edges contains H as a subgraph.*

We refer to $ex(H, n)$ as the *extremal number* or *extremal function* of H. Mantel [26] determined $ex(K_3, n)$ in 1907 and Turán [31] determined $ex(K_t, n)$ for all $t \geq 3$ in 1941, a result considered by many to mark the start of modern extremal graph theory. Outside of these results, the exact value of the extremal function is known for very few graphs (cf. [2, 4, 8]). In 1966, however, Erdős and Simonovits [11] extended previous work of Erdős and Stone [12] and determined $ex(H, n)$ asymptotically for arbitrary H. More precisely, this seminal theorem gives exact asymptotics for $ex(H, n)$ when H is a nonbipartite graph.

Theorem 1 (The Erdős-Stone-Simonovits Theorem)**.** *If H is a graph with chromatic number $\chi(H) \geq 2$, then*

$$ex(H, n) = \left(1 - \frac{1}{\chi(H) - 1}\right)\binom{n}{2} + o(n^2).$$

Given a family $\mathcal{F}$ of graphs, a graphic sequence π is *potentially $\mathcal{F}$-graphic* if there is a realization of π that contains some $F \in \mathcal{F}$ as a subgraph. If $\mathcal{F} = \{H\}$, we say that π is potentially H-graphic. The focus of this paper is the following problem posed by Erdős, Jacobson and Lehel in 1991 [10].

Problem 2. *Given a graph H, determine $\sigma(H, n)$, the minimum even integer such that every n-term graphic sequence π with $\sigma(\pi) \geq \sigma(H, n)$ is potentially H-graphic.*

We will refer to $\sigma(H, n)$ as the *potential number* or *potential function* of H. As $\sigma(\pi)$ is twice the number of edges in any realization of π, the Erdős-Jacobson-Lehel problem can be viewed as a potential degree sequence relaxation of the Turán problem.

In [10], Erdős, Jacobson and Lehel conjectured that $\sigma(K_t, n) = (t-2)(2n-t+1)+2$. The cases $t = 3, 4$ and 5 were proved separately (see respectively [10], [15] and [20], and [21]), and Li, Song and Luo [22] proved the conjecture true for $t \geq 6$ and $n \geq \binom{t}{2} + 3$. In addition to these results for complete graphs, the value of $\sigma(H, n)$ has been determined exactly for a number of other specific graph families, including complete bipartite graphs [5, 24], disjoint unions of cliques [13], and the class of graphs with independence number two [14] (for a number of additional examples, we refer the reader to the references of [14]). Despite this, relatively little is known in general about the potential function for arbitrary H. In this paper, we determine $\sigma(H, n)$ asymptotically for all H, thereby giving a potentially H-graphic sequence analogue to the Erdős-Stone-Simonovits Theorem.

2 Constructions and Statement of Main Result

We assume that H is an arbitrary graph of order k with at least one nontrivial component and furthermore that n is sufficiently large relative to k. We let $F < H$ denote that F is an induced subgraph of H. For each $i \in \{\alpha(H)+1, \ldots, k\}$ let

$$\nabla_i(H) = \min\{\Delta(F) : F < H, |V(F)| = i\},$$

and consider the sequence

$$\widetilde{\pi}_i(H, n) = ((n-1)^{k-i}, (k-i+\nabla_i(H)-1)^{n-k+i}).$$

If this sequence is not graphic, that is if $n-k+i$ and $\nabla_i(H)-1$ are both odd, we reduce the smallest term by one. To see that this yields a graphic sequence, we make two observations. First, $(\nabla_i(H)-1)$-regular graphs of order $n-k+i \geq \nabla_i(H)$ exist whenever $\nabla_i(H)-1$ and $n-k+i$ are not both odd. If $n-k+i$ and $\nabla_i(H)-1$ are both odd, it is not difficult to show that the sequence $((\nabla_i(H)-1)^{n-k+i-1}, \nabla_i(H)-2)$ is graphic.

We first show that the sequence $\widetilde{\pi}_i(H, n)$ is not potentially H-graphic for each $i \in \{\alpha(H)+1, \ldots, k\}$, thus establishing a lower bound on $\sigma(H, n)$.

Proposition 1. *If H is a graph of order k and n is a positive integer, then $\sigma(H, n) \geq \sigma(\widetilde{\pi}_i(H, n)) + 2$ for all $i \in \{\alpha(H)+1, \ldots, k\}$.*

Proof. Every realization G of $\widetilde{\pi}_i(H,n)$ is a complete graph on $k-i$ vertices joined to an $(n-k+i)$-vertex graph G_i with maximum degree $\nabla_i - 1$. Any k-vertex subgraph of G contains at least i vertices in G_i. Thus H is not a subgraph of G since every i-vertex induced subgraph of H has maximum degree at least ∇_i. $\square$

The focus of this paper is the asymptotic behavior of the potential function. As such, let

$$\widetilde{\sigma}_i(H) = 2(k-i) + \nabla_i(H) - 1,$$

which is the leading coefficient of $\sigma(\widetilde{\pi}_i(H,n))$. In [14], the first author and J. Schmitt conjectured the following.

Conjecture 1. *Let H be a graph, and let $\epsilon > 0$. There exists an $n_0 = n_0(\epsilon, H)$ such that for any $n > n_0$,*

$$\sigma(H,n) \le \max_{H' \subseteq H} (\widetilde{\sigma}_{\alpha(H')+1}(H') + \epsilon)n.$$

The condition that one must examine subgraphs of H is necessary. As an example, for $t \ge 3$ let H be obtained by subdividing one edge of $K_{1,t}$. Since $K_{1,t}$ is a subgraph of H, any sequence that is potentially H-graphic is necessarily potentially $K_{1,t}$-graphic. However, both graphs have independence number t and $\widetilde{\sigma}_{t+1}(H) < \widetilde{\sigma}_{t+1}(K_{1,t})$.

We show that in fact one needs only examine somewhat large induced subgraphs of H. The following, which determines the asymptotics of the potential function precisely for arbitrary H, is the main result of this paper.

Theorem 2. *Let H be a graph of order k and let n be a positive integer. If $\widetilde{\sigma}(H)$ is the maximum of $\widetilde{\sigma}_i(H)$ for $i \in \{\alpha(H)+1, \ldots, k\}$, then*

$$\sigma(H,n) = \widetilde{\sigma}(H)n + o(n).$$

As was pointed out in [14], Conjecture 1 is correct for all graphs H for which $\sigma(H,n)$ is known. Consequently, it is feasible that Theorem 2 is actually an affirmation of Conjecture 1. That is, for all H is it possible that

$$\widetilde{\sigma}(H) = \max_{H' \subseteq H} (\widetilde{\sigma}_{\alpha(H')+1}(H')),$$

however we are unable to either verify or disprove this at this time.

The proof of Theorem 2 is an immediate consequence of Proposition 1 and the following result.

Theorem 3. *Let H be a graph, and let $\omega = \omega(n)$ be an increasing function that tends to infinity with n. There exists an $N = N(\omega, H)$ such that for any $n \ge N$,*

$$\sigma(H,n) \le \widetilde{\sigma}(H)n + \omega(n).$$

The proof of Theorem 3 relies on repeated use of the following theorem, which may be of independent interest. Let H be a graph, and let $(h_1, \ldots, h_k)$ be the degree sequence of H. A graphic sequence $\pi = (d_1, \ldots, d_n)$ is *degree sufficient* for H if $d_i \ge h_i$ for all $i \in \{1, \ldots, k\}$.

Theorem 4 (The Bounded Maximum Degree Theorem)**.** *Let H be a graph of order k and let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence with n sufficiently large satisfying the following:*

1. *π is degree sufficient for H, and*
2. *$d_n \geq k - \alpha(H)$.*

There exists a function $f = f(\alpha(H), k)$ such that if $d_1 < n - f(\alpha(H), k)$, then π is potentially H-graphic.

In Section 3 we present several technical lemmas used in the proofs of Theorems 3 and 4. In Section 4 we prove the Bounded Maximum Degree Theorem, with the proof of Theorems 1 and 3 following in Section 5.

3 Technical Lemmas

We will need the following results from [23] and [32].

Theorem 5 (Yin and Li)**.** *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing graphic sequence and let k be a positive integer. If $d_k \geq k - 1$ and $d_i \geq 2(k-1) - i$ for all $i \in \{1, \ldots, k-1\}$, then π is potentially K_k-graphic.*

We let $G \vee H$ denote the standard join of G and H.

Lemma 1 (Yin)**.** *If π is a potentially $K_r \vee \overline{K_s}$-graphic sequence, then there is a realization of π in which the vertices in the copy of K_r are the r vertices of highest degree and the vertices in the copy of $\overline{K_s}$ are the next s vertices of highest degree.*

We now present a classical result of Kleitman and Wang [19] that generalizes the Havel-Hakimi algorithm [16, 18].

Theorem 6 (Kleitman and Wang)**.** *Let $\pi = (d_1, \ldots, d_n)$ be a nonincreasing sequence of nonnegative integers. If π_i is the sequence defined by*

$$\pi_i = \begin{cases} (d_1 - 1, \ldots, d_{d_i} - 1, d_{d_i+1}, \ldots, d_{i-1}, d_{i+1}, \ldots, d_n) & if\ d_i < i \\ (d_1 - 1, \ldots, d_{i-1} - 1, d_{i+1} - 1, \ldots, d_{d_i+1} - 1, d_{d_i+2}, \ldots, d_n) & if\ d_i \geq i, \end{cases}$$

then π is graphic if and only if π_i is graphic.

The process of removing a term from a graphic sequence as described in the Kleitman-Wang algorithm is referred to as *laying off* the term d_i from π, and the sequence π_i obtained is frequently called a *residual* sequence. Iteratively applying the Kleitman-Wang algorithm yields the following, which generalizes related lemmas from [3, 13].

Lemma 2. *Let m and k be positive integers and let $\omega = \omega(n)$ be an increasing function that tends to infinity with n. There exists $N = N(m, k, \omega)$ such that for all $n \geq N$, if π is an n-term graphic sequence such that $\sigma(\pi) \geq mn + \omega(n)$, then*

1. *π is potentially K_k-graphic, or*
2. *iteratively applying the Kleitman-Wang algorithm by laying off the minimum term in each successive sequence will eventually yield a graphic sequence π' with n' terms satisfying the following properties: (a) $\sigma(\pi') \geq mn' + \omega(n')$, (b) the minimum term in π' is greater than $m/2$, and (c) $n' \geq \frac{\omega(n)}{2(k-2)-m}$.*

Proof. First observe that if $k \leq 2$, then the result is trivial, so we assume that $k \geq 3$. Let $\pi = \pi_0$. If the minimum term of π_i is at least $m/2$, let $\pi' = \pi_i$. Otherwise, obtain the $n-i-1$-term graphic sequence π_{i+1} by applying the Kleitman-Wang algorithm to lay off a minimum term from π_i and then sorting the resulting sequence in nonincreasing order. Note that always $\sigma(\pi_{i+1}) \geq \sigma(\pi_i) - m$, and by induction we have $\sigma(\pi_{i+1}) \geq m(n-i-1) + \omega(n)$.

A consequence of the Kleitman-Wang algorithm is that if π_i is potentially K_k-graphic, then π_0 is potentially K_k-graphic. By the results of [10], [15] and [20], [21], and [22] we know that $\sigma(K_k, n) = 2(k-2)n-(k-1)(k-2)+2$ for $n \geq \binom{k}{2}+3$. Thus, if $\sigma(\pi_i) \geq 2(k-2)(n-i)+2$ for some $i \geq 0$, then π is potentially K_k-graphic. Considering the case when $i = 0$, we may assume that $m < 2(k-2)$. If $i \geq n - \frac{\omega(n)}{2(k-2)-m}$, then

$$\begin{aligned}\sigma(\pi_i) - 2(k-2)(n-i) &\geq (m-2(k-2))(n-i) + \omega(n) \\ &\geq (m-2(k-2))\left(\frac{\omega(n)}{2(k-2)-m}\right) + \omega(n) \\ &\geq 0.\end{aligned}$$

Therefore, if $i \geq n - \frac{\omega(n)}{2(k-2)-m}$ and $\frac{\omega(n)}{2(k-2)-m} \geq \binom{k}{2} + 3$, then π_i is potentially K_k-graphic and consequently π is potentially K_k-graphic.

It now follows that if π is not potentially K_k-graphic, then the process of applying the Kleitman-Wang algorithm must output $\pi' = \pi_i$ for some $i < n - \frac{\omega(n)}{2(k-2)-m}$. Since ω is increasing and $n' \leq n$, it follows that $\sigma(\pi') \geq mn' + \omega(n')$. □

As demonstrated in Lemma 2, repeatedly laying off minimum terms from a sequence via the Kleitman-Wang algorithm can allow us to obtain a residual sequence that is denser than π and also has a large minimum degree, which may facilitate the construction of a realization that contains H. At times, we may instead wish to delete vertices from specific realizations of π. The next lemma, the proof of which is essentially identical to that of the necessity of the Havel-Hakimi and Kleitman-Wang algorithms, demonstrates that this can a viable approach.

Lemma 3. *Let G be a graph with degree sequence π and let $v \in V(G)$. If $\pi(G-v)$ is potentially H-graphic, then π is potentially H-graphic. In particular, if π_i is any residual sequence obtained from π via the Kleitman-Wang algorithm and π_i is potentially H-graphic, then π is potentially H-graphic.*

Proof. Let $\pi = (d_1, \ldots, d_n)$ and let $V(G) = \{v_1, \ldots, v_n\}$ such that $d(v_i) = d_i$. Assume that π_i is potentially H-graphic and let G' be a realization of $\pi(G - v_i)$ that contains H as a subgraph. If $d_i < i$, obtain a realization of π containing H by joining v_i to vertices with degrees $d_1 - 1, \ldots, d_{d_i} - 1$ in G'. If $d_i \geq i$, obtain a realization of π containing H by joining v_i to vertices with degrees $d_1 - 1, \ldots, d_{i-1} - 1, d_{i+1} - 1, \ldots, d_{d_i+1} - 1$ in G'. □

In the same spirit, we also give the following useful lemma. As the proof is straightforward, we omit it here. For a graph G and an integer i with $i \in \{0, \ldots, |V(G)|\}$, let $\mathcal{D}^{(i)}(G)$ denote the family of graphs obtained by deleting exactly i vertices from G or, in other words, the family of induced subgraphs of G with order $|V(G)| - i$.

Lemma 4. *If $\pi = (n-1, d_2, \ldots, d_n)$ is a nonincreasing graphic sequence, then π is potentially H-graphic if and only if $\pi_1 = (d_2 - 1, \ldots, d_n - 1)$ is potentially H'-graphic for some H' in $\mathcal{D}^{(1)}$.*

Finally, for completeness, we state the classic characterization of graphic sequences due to Erdős and Gallai [9], which will be useful in the proof of Theorem 3.

Theorem 7 (The Erdős-Gallai Graphicality Criteria)**.** *If $\pi = (d_1, \ldots, d_n)$ is a nonincreasing sequence of nonnegative integers, then π is graphic if and only if*

$$\sum_{j=1}^{t} d_j \leq \binom{t}{2} + \sum_{j=t_1}^{n} \min\{d_j, t\}$$

for all $t \in \{1, \ldots, n-1\}$.

4 Proof of the Bounded Maximum Degree Theorem

In this section, we prove the Bounded Maximum Degree Theorem (Theorem 4). Both here and in the proof of Theorem 3, we demonstrate the existence of a realization of π that either contains H or some supergraph of H, for instance K_k or $K_{k-\alpha(H)} \vee \overline{K}_{\alpha(H)}$.

Let $e_1 = u_1v_1$ and $e_2 = u_2v_2$ be edges in a graph G such that u_1u_2 and v_1v_2 are not in $E(G)$. Removing e_1 and e_2 from G and replacing them with the edges u_1u_2 and v_1v_2 results in a graph with the same degree sequence as G. This operation is called a *2-switch*, but throughout the literature on degree sequences has also been referred to as a *swap*, *rewiring* or *infusion*. In [27], Petersen showed that, given realizations G_1 and G_2 of a graphic sequence π, G_1 can be obtained from G_2 via a sequence of 2-switches. Both here and in Section 5, we will utilize both 2-switches and more general edge-exchange operations. Specifically, a list of vertices $C = v_0, v_1, \ldots, v_t$ with $v_t = v_0$ in a graph G is an *alternating circuit* if $v_{i-1}v_i \in E(G)$ if and only if $v_iv_{i+1} \notin E(G)$. Observe that removing the edges in $E(C) \cap E(G)$ from G and adding the edges in $E(C) \cap E(\overline{G})$ to G preserves the degree sequence of G.

Proof. Let $V(H) = \{u_1, \ldots, u_k\}$, indexed such that $d_H(u_i) \geq d_H(u_j)$ when $i \leq j$. Let us assume that π satisfies the hypothesis of the theorem, but is not potentially H-graphic. In a realization G of π, let $S = \{v_1, \ldots, v_k\}$ be a set of vertices such that $d_G(v_i) = d_i$ and let H_S be the graph with vertex set S in which two vertices v_i, v_j are adjacent if and only $u_iu_j \in E(H)$. If all edges of H_S are edges of G, then H_S is a subgraph of G isomorphic to H and π is potentially H-graphic. Hence, let us assume that G is a realization which maximizes $|E(G) \cap E(H_S)|$, but that the edge $v_iv_j \in E(H_S)$ while $v_iv_j \notin E(G)$. Now, because $d_G(v_i) \geq d_H(u_i) = d_{H_S}(v_i)$ and $d_G(v_j) \geq d_H(u_j) = d_{H_S}(v_j)$, there must exist (not necessarily distinct) vertices a_i and a_j such that $v_ia_i, v_ja_j \in E(G)$, while $v_ia_i, v_ja_j \notin E(H_S)$.

We begin by showing that many vertices of $V(G) - S$ have neighbors in $V(G) - S$. Specifically, we claim that there are at most

$$g(\alpha(H), k) = \binom{k}{k - \alpha(H)} \left[2\binom{k - \alpha(H)}{2} + \alpha(H) - 1 \right]$$

vertices w in $V(G) - S$ such that $N(w) \subseteq S$.

Indeed, let us assume that there are at least $g(\alpha(H), k) + 1$ vertices w in $V(G) - S$ such that $N(w) \subseteq S$. By the hypothesis of the theorem, each vertex $w \in V(G)$ satisfies $d_G(w) \geq k - \alpha(H)$. For each vertex w such that $N(w) \subseteq S$, let S_w be a set of $k - \alpha(H)$ vertices such that $S_w \subseteq N(w)$. Let $\ell = 2\binom{k-\alpha(H)}{2} + \alpha(H)$. By the pigeonhole principle, there exists an independent set $\{w_1, \ldots, w_\ell\}$ of vertices and a $k - \alpha(H)$ element subset $\widehat{S}$ of S such that $S_{w_i} = \widehat{S}$ for each $1 \leq i \leq \ell$.

Let $\mathcal{P}$ be a family of $\binom{k-\alpha(H)}{2}$ disjoint pairs of vertices from $\{w_1, \ldots, w_{\ell-\alpha(H)}\}$, and to each pair $\{v_r, v_s\}$ of vertices in $\widehat{S}$ associate a distinct pair $P(r, s) \in \mathcal{P}$. If v_rv_s is not an edge in G and we suppose $P(r, s) = \{w_{r'}.w_{s'}\}$, then we perform the 2-switch that replaces the edges $v_rw_{r'}$ and $v_sw_{s'}$ with the non-edges v_rv_s and $w_{r'}w_{s'}$. In this way arrive at a realization of π in which the graph induced by $\widehat{S}$ is complete, while maintaining the property that the vertices of $\widehat{S}$ are joined to $\{w_{\ell-\alpha(H)+1}, \ldots, w_\ell\}$. This produces a realization of π that contains $K_{k-\alpha(H)} \vee \overline{K_{\alpha(H)}}$ as a subgraph, contradicting the assumption that π is not potentially H-graphic. Consequently, we may assume that there are at most $g(\alpha(H), k)$ vertices w in $V(G) - S$ such that $N(w) \subseteq S$.

We will now use the fact that many vertices of $V(G) - S$ have neighbors in $V(G) - S$ to exhibit an edge-exchange that inserts the edge v_iv_j into G at the expense of the edges v_ia_i and v_ja_j while preserving each edge in H_S. Let

$$f(\alpha(H), k) = [g(\alpha(H), k) + 4k^2] + k + 1,$$

and let $d_1 \leq n - 1 - f(\alpha(H), k)$. If we let

$$X_i = \{v \in (V(G) - S) - N_{G-S}(a_i) \ : \ d_{G-S}(v) > 0\}$$

and

$$X_j = \{v \in (V(G) - S) - N_{G-S}(a_j) \ : \ d_{G-S}(v) > 0\},$$

then, as at most $g(\alpha, k)$ vertices have their neighborhoods entirely contained in S, it follows that

$$|X_i|, |X_j| \geq 4k^2.$$

Let $Y_i = N_{G-S}(X_i)$ and $Y_j = N_{G-S}(X_j)$.

By assumption, π is not potentially H-graphic, and is therefore not potentially K_k-graphic. As such, Theorem 5 implies that each vertex in $G-S$ has degree at most $2k-4 < 2k$. Let y_i be a vertex in Y_i, and let x_i be a neighbor of y_i in X_i. There are at least $4k^2-1$ vertices in X_j that are not x_i, and at least $4k^2-1-(2k-5)$ of these vertices have a neighbor in Y_j that is not y_i. Since vertices in $V(G)-S$ have degree at most $2k-4$, there are more than $(4k^2-2k+4)/(2k) = 2k-1+2/k$ vertices in Y_j that are not y_i and furthermore have a neighbor in X_j that is not x_i. Since $d(y_i) < 2k$, there exists a vertex y_j, distinct from y_i such that $y_iy_j \notin E(G)$ and y_j has a neighbor $x_j \in X_j$ that is distinct from x_i. In this case, exchanging the edges v_ia_i, v_ja_j, x_iy_i and x_jy_j for the nonedges v_iv_j, a_ix_i, a_jx_j and y_iy_j yields a realization of π that contradicts the maximality of G.

□

5 Proof of Theorem 3

The proof of Theorem 3 proceeds in two stages, each of which involves the execution of several key steps. Prior to presenting the full proof, we will outline these crucial steps to give a clearer picture of the structure and philosophy of the proof. Starting from π, we construct graphic sequences $\pi_0, \pi_1, \dots, \pi_\ell$ for some $\ell \le k-\alpha(H)$.

Each $\pi_i = (d_1^{(i)}, \dots, d_{n_i}^{(i)})$, which we assume to be nonincreasing and sufficiently long, will satisfy the following conditions:

(a) $d_{n_i}^{(i)} \ge k-i-\alpha(H)$, and

(b) if π_i is potentially $\mathcal{D}^{(i)}(H)$-graphic, then π is potentially H-graphic.

To initialize Stage 1, we apply Lemma 2 to π to obtain an n_0-term sequence π_0 with minimum term at least $k-\alpha(H)$. Stage 1 then proceeds as follows:

1. If $i = k-\alpha(H)$ or if $d_1^{(i)} < n_i - \binom{k}{\lceil k/2 \rceil}(k^2-1)$, then terminate Stage 1 and proceed to Stage 2 with $\ell = i$. Otherwise, $d_1^{(i)} \ge n_i - \binom{k}{\lceil k/2 \rceil}(k^2-1)$.
2. Let π_i' be the degree sequence of the graph obtained by deleting all nonneighbors of a vertex of maximum degree from some realization of π_i. If we let n_i' denote the number of terms in π_i', then the largest term in π_i' is $n_i'-1$.
3. Iteratively lay off smallest terms to obtain, via Lemma 2, a sequence π_i'' that has minimum term at least $k-i-\alpha(H)$. If there are n_i'' terms in π_i'', then the largest term in π_i'' will be $n_i''-1$.
4. Obtain π_{i+1} by laying off the largest term in π_i''.

Note that Condition (a) ensures that π_i satisfies Condition 2 of the Bounded Maximum Degree Theorem for any $(k-i)$-vertex graph H_i with $\alpha(H_i) \ge \alpha(H)$. The execution of Step 3 ensures that π_{i+1} will also satisfy Condition (a), highlighting the importance of Lemma 2 in the proof. Also note that even though Lemma 2 may require laying off a large fraction of the terms in π_i', we still have that $n_{i+1} \to \infty$ provided $n_i \to \infty$. Thus we may assume that n_{i+1} is sufficiently large.

Further, given Lemma 4, we conceptualize the terms laid off in the first $i+1$ iterations Step 4 as forming a clique of order $i+1$ that is joined to any realization of π_{i+1}. Hence, it suffices to show that π_{i+1} is potentially $\mathcal{D}^{(i+1)}$-graphic.

More specifically, we construct π_i'' by deleting vertices from a realization of π (Step 2) and/or laying off terms from π_i' (Step 3). Hence, by Lemma 3, if π_i'' is potentially H-graphic, then π_i is. Further, by Lemma 4, π_{i+1} is potentially $\mathcal{D}^{(i+1)}(H)$-graphic if and only if π_i'' is potentially $\mathcal{D}^{(i)}(H)$-graphic. The assumption that π is potentially H-graphic if π_i is potentially $\mathcal{D}^{(i)}(H)$-graphic then ensures that π is potentially H-graphic provided that π_{i+1} is potentially $\mathcal{D}^{(i+1)}(H)$-graphic, satisfying Condition (b).

In Stage 2 of the proof, we use edge-exchanges to demonstrate that π_ℓ is potentially $\mathcal{D}^{(\ell)}(H)$-graphic. First we use the Bounded Maximum Degree Theorem to obtain a realization of π_ℓ that contains the complete split graph $H_\ell = K_{k-\ell-q} \vee \overline{K}_q$ for some $q \geq \alpha(H)$. Observe that the bound on $d_1^{(\ell)}$ from Step 1 of Stage 1 is chosen to be low enough for the Bounded Maximum Degree Theorem to apply for any choice of q. Then, a careful analysis of Steps 1-4 shows that π_ℓ, which we know is sufficiently long, is also sufficiently dense in the following sense. If we choose H_ℓ such that q is minimum, then we can show that the vertices in the independent set of order q in H_ℓ all must have degree at least $k-\ell-q+\nabla_q$ in the realization of π_ℓ. From there it is possible to find a realization of π_ℓ that contains $K_{k-\ell-q} \vee F_q$, where F_q is a q-vertex induced subgraph of H with maximum degree ∇_q. As $K_{k-\ell-q} \vee F_q$ is a supergraph of some graph in $\mathcal{D}^{(\ell)}(H)$, the result follows.

We are now ready to give the proof of Theorem 3 in full detail.

Proof. Let π be an n-term graphic sequence with $n > N(\omega, H)$ such that $\sigma(\pi) \geq \widetilde{\sigma}(H)n + \omega(n)$. It suffices to show that π is potentially H-graphic. If π is potentially K_k-graphic, then the result is trivial. Therefore, by Theorem 5, we may assume that $d_{k+1} \leq 2k-4$. Furthermore, throughout the proof we may assume that when applying Lemma 2 to π or a residual sequence obtained from π, the resulting sequence satisfies Conclusion (2) of that lemma.

To initialize Stage 1, apply Lemma 2 to π to obtain an n_0-term graphic sequence $\pi_0 = (d_1^{(0)}, \ldots, d_{n_0}^{(0)})$ with minimum entry at least $\widetilde{\sigma}(H)/2$ and $\sigma(\pi_0) \geq \widetilde{\sigma}(H)n_0 + \omega(n_0)$.

Starting with this π_0, we iteratively construct sequences $\pi_i = (d_1^{(i)}, \ldots, d_{n_i}^{(i)})$ for $i \geq 1$ that satisfy the criteria given in the outline above:

(a) $d_{n_i}^{(i)} \geq k - i - \alpha(H)$, and

(b) if π_i is potentially $\mathcal{D}^{(i)}(H)$-graphic, then π is potentially H-graphic.

The process terminates and we set $\ell = i$ if either $i = k - \alpha(H)$ or $d_1^{(i)} < n_i - \binom{k}{\lceil k/2 \rceil}(k^2-1)$. Otherwise, let $\widehat{G}_i$ be a realization of π_i in which a vertex of degree $d_1^{(i)}$, which we call $v_1^{(i)}$, is adjacent to the next $d_1^{(i)}$ vertices of highest degree; such a realization exists as a consequence of the Kleitman-Wang algorithm. From $\widehat{G}_i$, obtain the graph $\widehat{G}'_i$ by deleting the nonneighbors of $v_1^{(i)}$ and let $\pi'_i = \pi(\widehat{G}'_i)$. Apply Lemma 2 to π'_i, and let π''_i be the graphic sequence from Conclusion (2) of that lemma. Finally, lay off the largest term from π''_i to obtain π_{i+1}.

We claim that $\sigma(\pi_i) \geq (\widetilde{\sigma}(H) - 2i)n_i + \omega(n_i)/2^i$ and the minimum term in π_i is at least $\widetilde{\sigma}(H)/2 - i \geq k - i - \alpha(H)$ for all $i \in \{0\ldots,\ell\}$. We use induction on i. First observe that $\sigma(\pi_0) \geq \widetilde{\sigma}(H)n_0 + \omega(n_0)$ and (as we assume that Conclusion (2) results from all applications of Lemma 2) the minimum term of π_0 is at least $\widetilde{\sigma}(H)/2 \geq k - \alpha(H)$. Let

$$M = 2\binom{k}{\lceil k/2\rceil}(k^2-1)(2k-4).$$

Creating π'_i entails deleting at most $\binom{k}{\lceil k/2\rceil}(k^2-1)$ vertices each with degree at most $2k-4$, and consequently, by induction,

$$\sigma(\pi'_i) \geq \sigma(\pi_i) - M \geq (\widetilde{\sigma}(H) - 2i)n_i + \omega(n_i)/2^i - M.$$

By assumption, n_i is sufficiently large, so we have $\omega(n_i)/2^i \geq 2M$, and therefore $\sigma(\pi'_i) \geq (\widetilde{\sigma}(H) - 2i)n_i + \omega(n_i)/2^{i+1}$. We next apply Lemma 2 to π'_1 to obtain a sequence π''_i with n''_i terms. It follows that

$$\sigma(\pi''_i) \geq (\widetilde{\sigma}(H) - 2i)n''_i + \omega(n''_i)/2^{i+1}$$

and we also have that the minimum term in π''_i is at least $k - i - \alpha(H)$. The maximum term in π''_i is $n''_i - 1$, and we lay off this term to obtain π_{i+1}. This yields

$$\begin{aligned}\sigma(\pi_{i+1}) &\geq (\widetilde{\sigma}(H) - 2i)n''_i + \omega(n''_i)/2^{i+1} - 2(n''_i - 1)\\ &\geq (\widetilde{\sigma}(H) - 2(i+1))n_{i+1} + \omega(n_{i+1})/2^{i+1}.\end{aligned}$$

Also, as the minimum term in π''_i is at least $k - i - \alpha(H)$ and decreases by 1 when the maximum term is laid off, we have that π_{i+1} satisfies Condition (a).

Next we show that Condition (b) holds, also via induction on i. If we let $\mathcal{D}^{(0)}(H) = \{H\}$, then the claim holds for $i = 0$. For some $i < \ell$, assume that if π_i is potentially $\mathcal{D}^{(i)}(H)$-graphic, then π is potentially H-graphic. By Lemma 3, if π'_i is potentially $\mathcal{D}^{(i)}(H)$-graphic, then so too is π_i. Similarly, Lemma 3 implies that if π''_i is potentially $\mathcal{D}^{(i)}(H)$-graphic, then π'_i is graphic. Finally, by Lemma 4, we have that π_{i+1} is potentially $\mathcal{D}^{(i+1)}(H)$-graphic if and only if π''_i is potentially $\mathcal{D}^{(i)}(H)$-graphic. Taken together, we get

$$\begin{aligned}\pi_{i+1} \text{ is potentially } \mathcal{D}^{(i+1)}(H)\text{-graphic} &\Longrightarrow \pi''_i \text{ is potentially } \mathcal{D}^{(i)}(H)\text{-graphic}\\ &\Longrightarrow \pi'_i \text{ is potentially } \mathcal{D}^{(i)}(H)\text{-graphic}\\ &\Longrightarrow \pi \text{ is potentially } H\text{-graphic},\end{aligned}$$

as desired. Thus, π_{i+1} satisfies Condition (b).

Therefore, it remains to show that π_ℓ is potentially $\mathcal{D}^{(\ell)}(H)$-graphic. If $\ell = k - \alpha(H)$, then $\mathcal{D}^{(\ell)}(H)$ contains the empty graph on $\alpha(H)$ vertices, and since π_ℓ is sufficiently long the result holds trivially. Thus we may assume that $\ell < k - \alpha(H)$ and that $d_1^{(\ell)} < n_\ell - \binom{k}{\lceil k/2\rceil}(k^2-1)$.

In π_ℓ, let $t = \max\{i : d_i^{(\ell)} \geq k-\ell-1\}$. First assume that $t \geq k-\ell-\alpha(H)$. In this case, π_ℓ is degree sufficient for $K_{k-\ell-\alpha(H)} \vee \overline{K_{\alpha(H)}}$. Furthermore, the minimum term in π_ℓ is at least $k-\ell-\alpha(H)$, and $d_1^{(\ell)} < n_\ell - \binom{k}{\lceil k/2 \rceil}(k^2-1) < n_\ell - f(\alpha(H), k-\ell)$. Thus the Bounded Maximum Degree Theorem implies that π_ℓ is potentially $K_{k-\ell-\alpha(H)} \vee \overline{K_{\alpha(H)}}$-graphic. Since $K_{k-\ell-\alpha(H)} \vee \overline{K_{\alpha(H)}}$ is a supergraph of a $(k-\ell)$-vertex subgraph of H, it follows that π_ℓ is potentially $\mathcal{D}^{(\ell)}(H)$-graphic.

Now assume that $t < k-\ell-\alpha(H)$. Let F_i denote an i-vertex induced subgraph of H that achieves $\Delta(F_i) = \nabla_i(H)$. We show that π_ℓ is degree sufficient for $K_t \vee F_{k-\ell-t}$. First observe that since $k-\ell-t > \alpha(H)$ we have $\widetilde{\sigma}(H) - 2\ell \geq \widetilde{\sigma}_{k-\ell-t}(H) - 2\ell = 2t - \nabla_{k-\ell-t}(H) - 1$. Thus

$$\sigma(\pi_\ell) \geq (2t - \nabla_{k-\ell-t}(H) - 1)n_\ell + \omega(n_\ell)/2^\ell.$$

However, if π_ℓ is not degree sufficient for $K_t \vee F_{k-\ell-t}$, then

$$\sigma(\pi_\ell) \leq t(n_\ell - 1) + (k-\ell-t-1)(k-\ell-2) + (n_\ell - k + l + 1)(t + \nabla_{k-\ell-t}(H) - 1)$$

$$< (2t + \nabla_{k-\ell-t}(H) - 1)n_\ell + \omega(n_\ell)/2^\ell$$

provided that n_ℓ is sufficiently large. Thus π_ℓ is degree sufficient for $K_t \vee F_{k-\ell-t}$. Note that $\alpha(F_{k-\ell-t})$ may be less than $\alpha(H)$. Therefore we are not able to immediately apply the Bounded Maximum Degree Theorem to obtain a realization of π_ℓ containing $K_t \vee F_{k-\ell-t}$.

However, in this case, π_ℓ is degree sufficient for $K_t \vee \overline{K_{k-\ell-t}}$. Since π_ℓ satisfies Condition (a), the minimum term of π_ℓ is at least $k-\ell-\alpha(H)$, and by assumption, $k-\ell-\alpha(H) > t$. Furthermore, $d_1^{(\ell)} < n_\ell - \binom{k}{\lceil k/2 \rceil}(k^2-1) < n_\ell - f(k-\ell-t, k-\ell)$. Thus by the Bounded Maximum Degree Theorem, there is a realization G_ℓ of π_ℓ containing $K_t \vee \overline{K_{k-\ell-t}}$. If $v_1, \ldots, v_{k-\ell}$ are the $k-\ell$ vertices of highest degree in G_ℓ, then by Lemma 1 we may assume that $\{v_1, \ldots, v_t\}$ is a clique that is completely joined to $\{v_{t+1}, \ldots, v_{k-\ell}\}$. Delete $v_1, \ldots, v_t$ from G_ℓ to obtain G'_ℓ, and let $\pi'_\ell = \pi(G'_\ell)$, with the order of π'_ℓ coming from the ordering of π_ℓ. Thus the first $k-\ell-t$ terms in π'_ℓ correspond to the vertices $\{v_{t+1}, \ldots, v_{k-\ell}\}$ in G_ℓ. Since the minimum degree of the vertices in G_ℓ is at least $k-\ell-\alpha(H)$ and $t < k-\ell-\alpha(H)$, the minimum term in π'_ℓ is at least 1. It remains to show that there is a realization of π'_ℓ that contains a copy of $F_{k-\ell-t}$ on the vertices $\{v_{t+1}, \ldots, v_{k-\ell}\}$.

To construct such a realization, place a copy of $F_{k-\ell-t}$ on the vertices $v_{t+1}, \ldots, v_{k-\ell}$. Since n_ℓ is sufficiently large, we may join any remaining edges incident to $\{v_{t+1}, \ldots, v_{k-\ell}\}$ to distinct vertices among the remaining $n_\ell - (k-\ell)$ vertices. It remains to show that there is a graph on the remaining $n_\ell - (k-\ell)$ vertices that realizes the residual sequence. This sequence has at least $n_\ell - (k-\ell) - (k-\ell-t)(k-\ell-3)$ positive terms with the maximum term being at most $k-\ell-2$. By Theorem 7, the Erdős-Gallai criteria, such a sequence is graphic provided that n_ℓ is sufficiently large. □

As noted above, Theorem 2 follows immediately from Theorems 1 and 3.

6 Conclusion

Having determined the asymptotic value of the potential function for general H, it may be of interest to study the structure of those n-term graphic sequences that are not potentially H-graphic, but whose sum is close to $\sigma(H, n)$. This line of inquiry would be related to recent work of Chudnovsky and Seymour [6], which for an arbitrary graphic sequence π gives a partial structural characterization of those graphic sequences π' for which no realization of π' contains *any* realization of π as an *induced* subgraph.

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